




MAGNANIMITER CRUCEM SUSTINE
GEORGE KENYON
of Peel Esqr.

TRACTATUS MATHEMATICUS

DE

Figurarum Curvilinearum

QUADRATURIS

ET

Locis Geometricis.

Autore

JOHANNE CRAIG.

LONDINI:

Proſtant apud *Sam. Smith* & *Benj. Walford*, Societatis Regiæ Typographos, ad Inſignia Principis, in Cœmeterio D. *Pauli*. M DC XCIII.

REVERENDO
In Christo Patri & Domino,
D^{no} GILBERTO,
Providintiâ divinâ, Episcopo *Sarisburiensi*, & Nobilissimi Ordinis à Periscelide
CANCELLARIO.

TRactatum hunc Geometricum tibi (Reverende Præsul) dicatum volui, ut Animi gratitudinem publicè testarer, ob innumeras Benevolentiæ tuæ notas toties & tam munificè in me collatas. Tu Studia nostra Consilio, & Sumptibus tuis promovisti; æquum proinde censeo, ut hoc qualecunque eorum specimen in lucem prodeat Nomini tuo consecratum. Neque tanta est Theologiam inter & Geometriam repugnantia, (quicquid Phanatici quidam utriusque pariter Ignari in contrarium obganniant) ut tibi res hujusmodi offerre absurdum videatur. Cognoscitur Deus ex operibus, ejusq; opera nemo, nisi Geometriâ adjutus, qui non vel prorsus ignorat, vel leviter admodum intelligit. Deus etenim in singulis Universi partibus formandis Geometram egit perfectissimam, hoc est, ratione operandi certissima & perfectissima semper & ubiq; usus est; quod tibi (Reverende Præsul) omnibusque interioris Philosophiæ peritis magis est manifestum, quam ut prolixè illud demonstrare necessè habeatur. Qui mirabilem Oculi fabricam bene per-

pendet, fatebitur utique Deum in illius formatione Leges Opticæ subtilissimas adhibuisse : quique notabilem Musculorum structuram, eorum concinnam cum ossibus unionem, ipsorumque Ossium elegantem compagem, necnon accuratam omnium Artuum connexionem (quibus Motus *Animalium peragitur) ritè intelligit ; proculdubio inveniet eximium hoc Corporis nostri Automaton fuisse juxta Regulas* Mechanicæ *Geometricas constructum.*

Possem multa alia Divinæ hujus Geometriæ egregia Exempla, ex cœlestium Corporum motibus, aliisque naturæ phænomenis petita proferre ; si opportunus daretur locus ea omnia minutatim enucleandi, quæ abundè demonstrarent Geometriam non minus sanæ Theologiæ, quam veræ Philosophiæ ancillari. Sed hæc consultò jam omitto, quorum veritas & evidentia te latere nequeunt, qui Philosophiam chymicis, aliisque experimentis, necnon sublimiores & magis reconditas Mathesios disciplinas tam accuratè prosequutus es, ut sine minima adulationis nota dicere jure liceat, vix aliquem majora auxilia ex solidis hisce fundamentis deduxisse, ad difficiliora Religionis problemata explicanda & stabilienda.

Plura jam non addo, ne officii mei immemor, ulterius quàm decet progrediar. Ut te Reipublicæ literariæ Decus, & florentissimæ Ecclesiæ Anglicanæ Ornamentum favore benigno protegat, & incolumem servet, Deum Opt. Max. supplex & ex Animo suo orat,

Servus Tuus Humillimus,

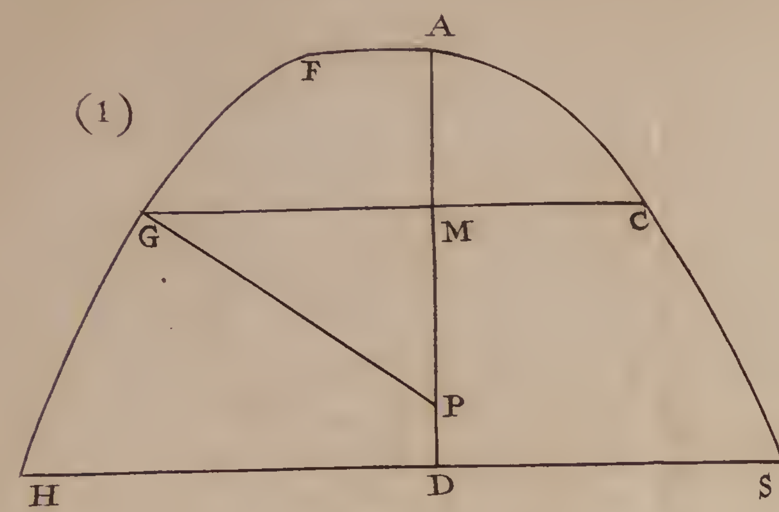
Tuique Observantissimus,

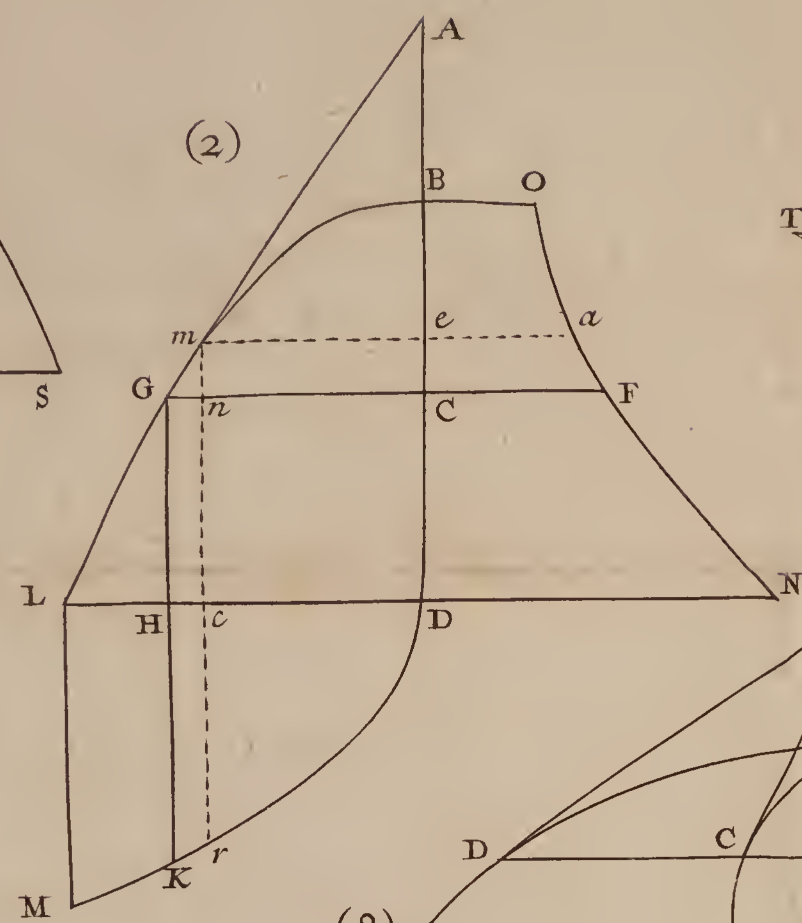
Jo. Craig.

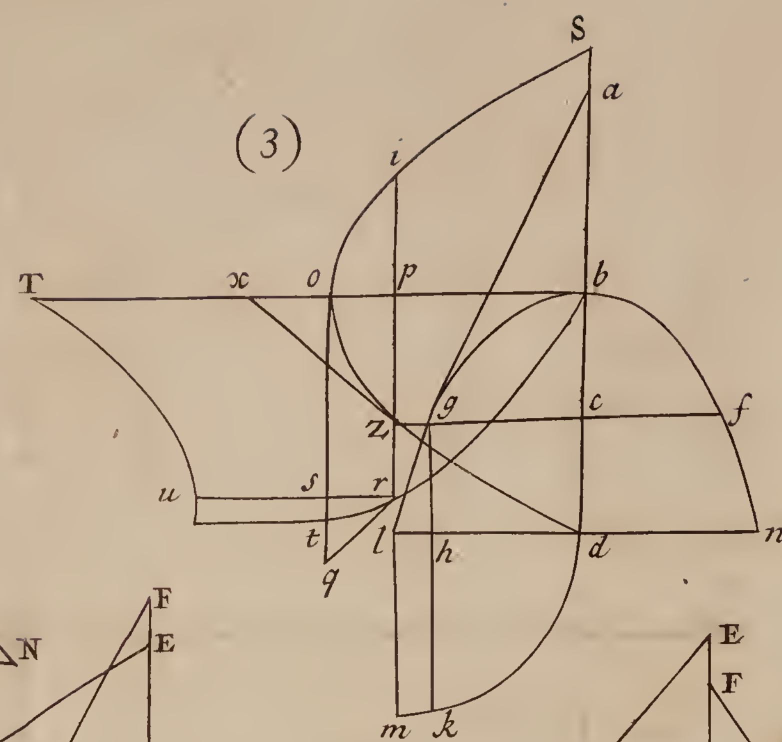


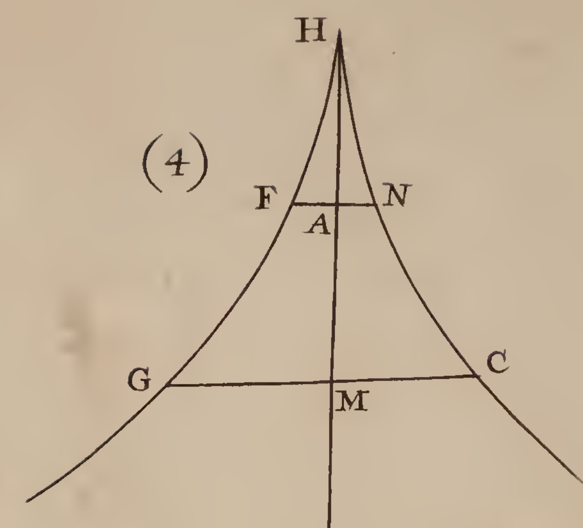


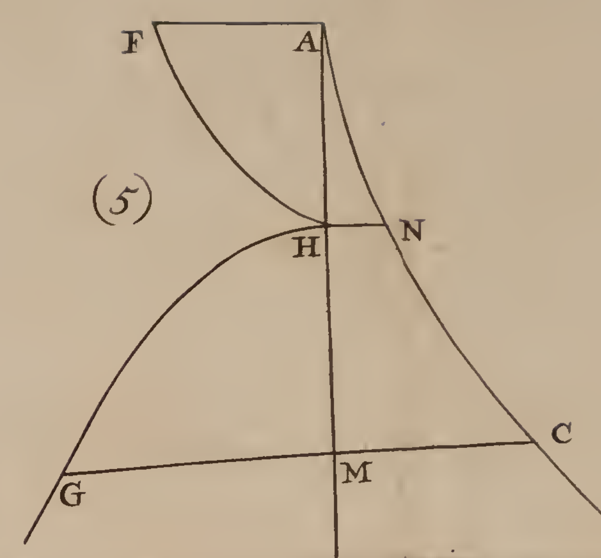


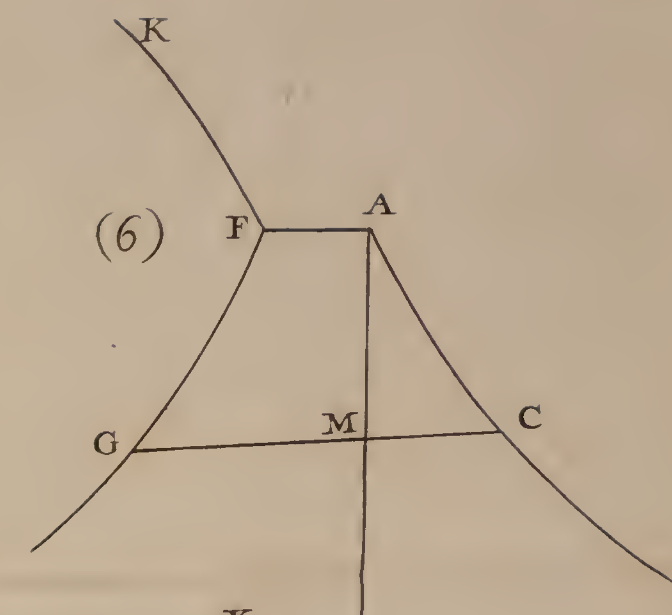

(1)
A F G M C P H D S

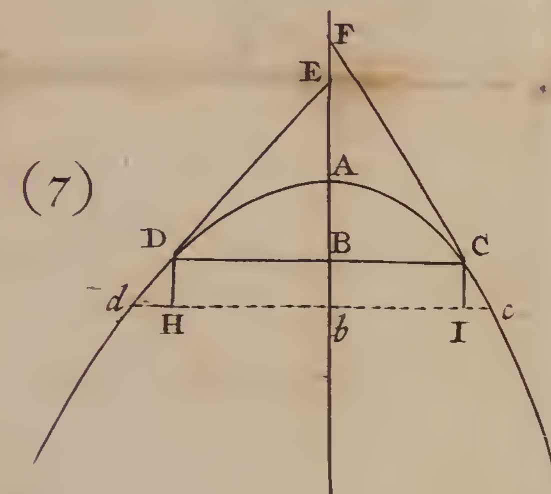

(2)
A B O m e a G n C F L H c D N M K r

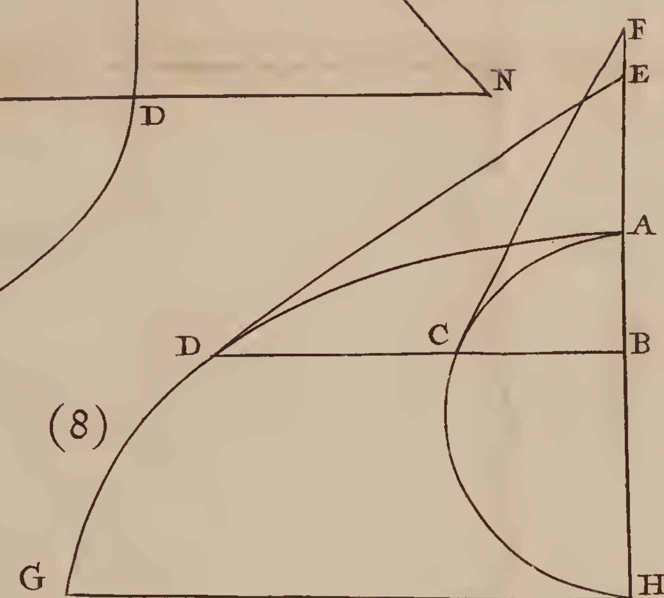

(3)
S a i T x o p b z g c f u s r t l h d n q m k

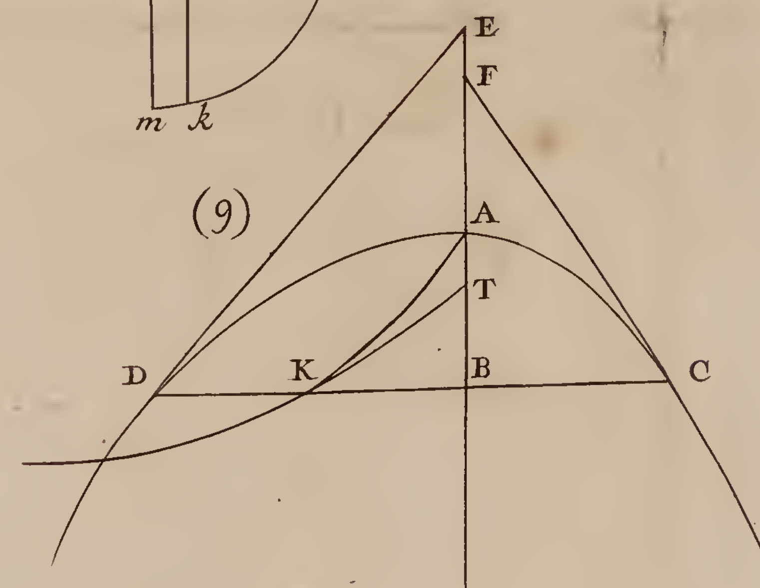

(4)
H F A N G M C

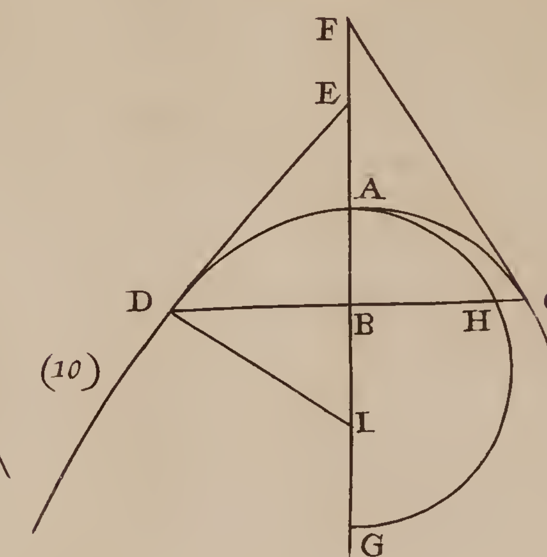

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F A H N G M C

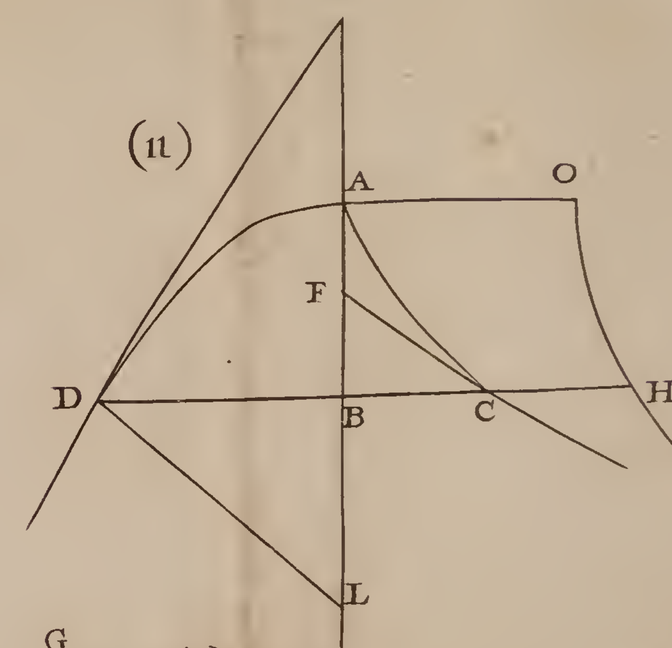

(6)
K F A G M C

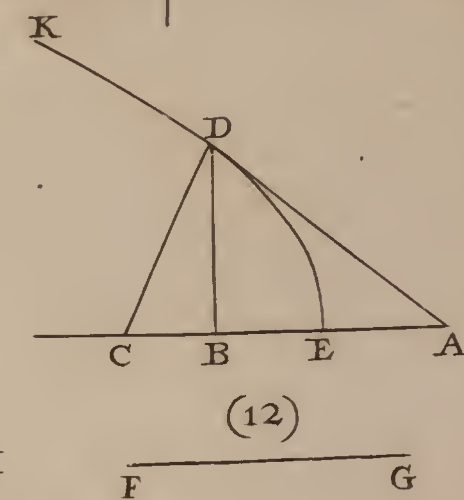

(7)
F E A D B C d H b I c

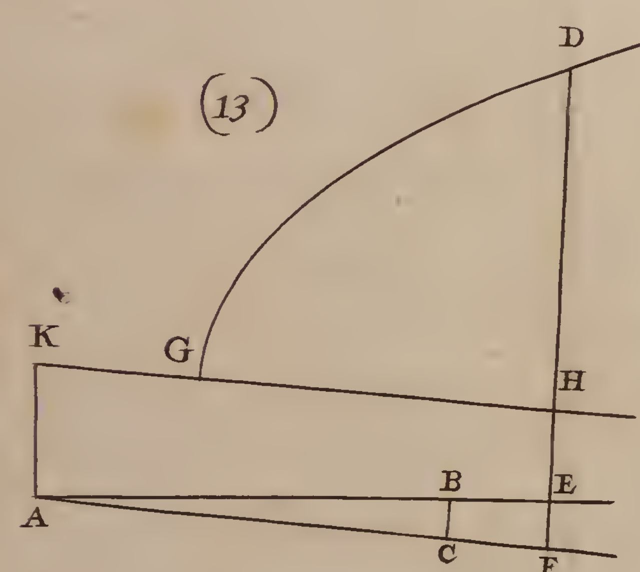

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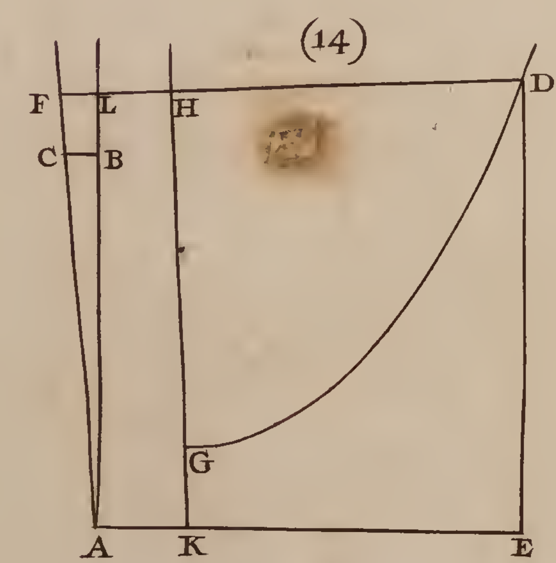

(9)
E F A T D K B C

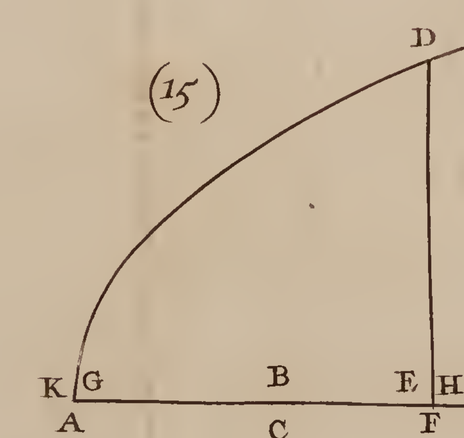

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F E A D B H C L G

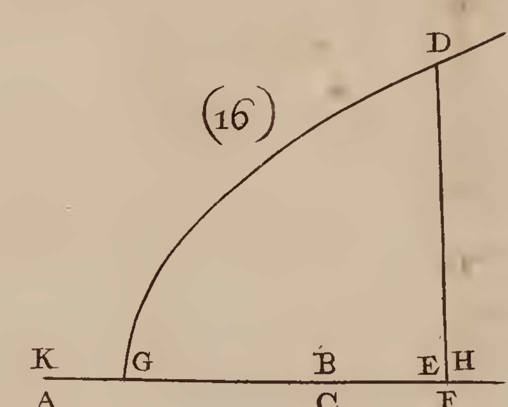

(11)
A O F D B C H L

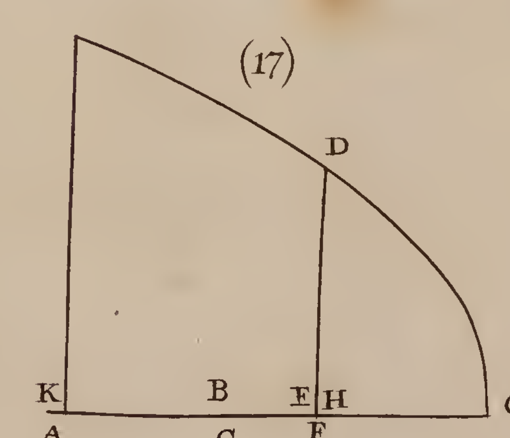

(12)
K D C B E A F G

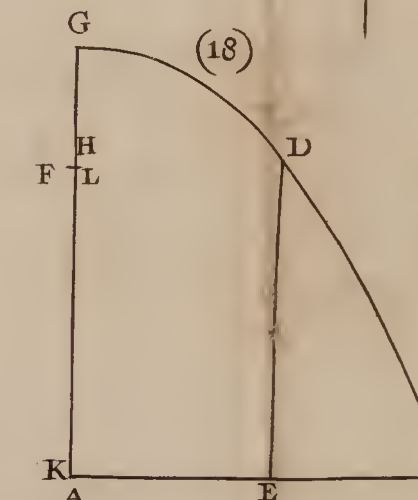

(13)
D K G H A B E C F

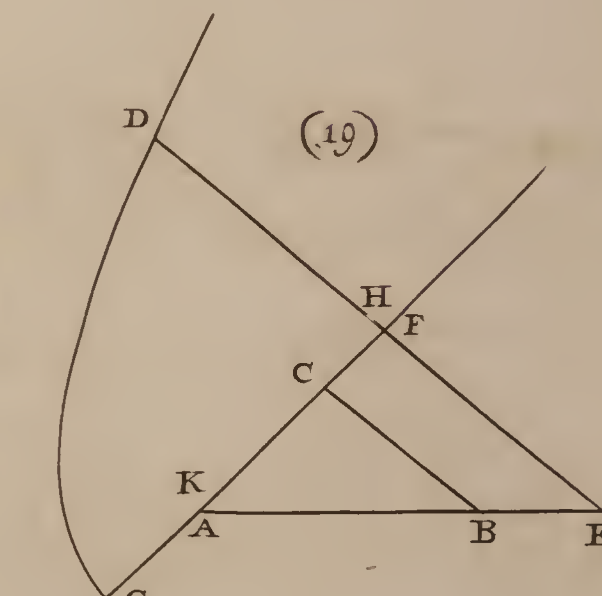

(14)
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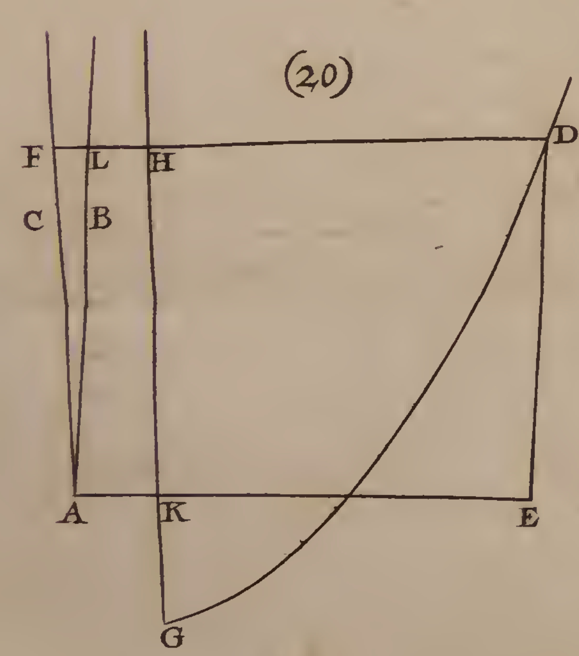

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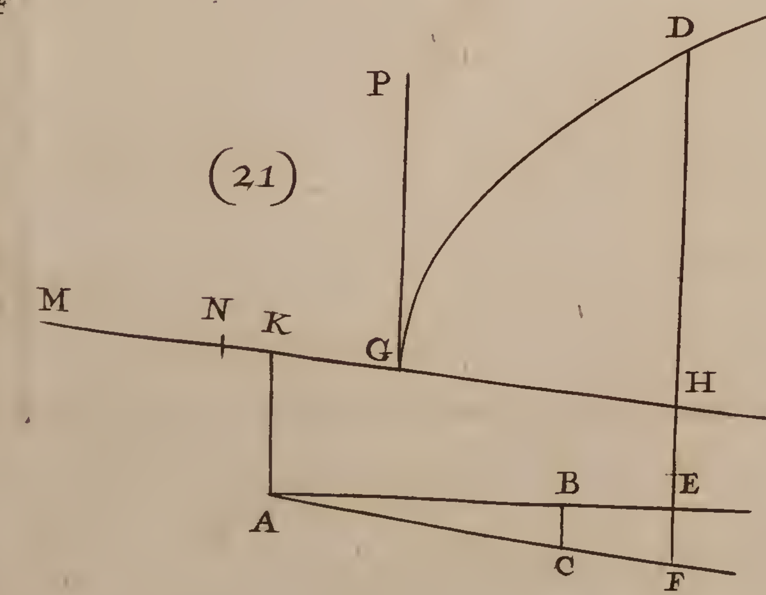

(16)
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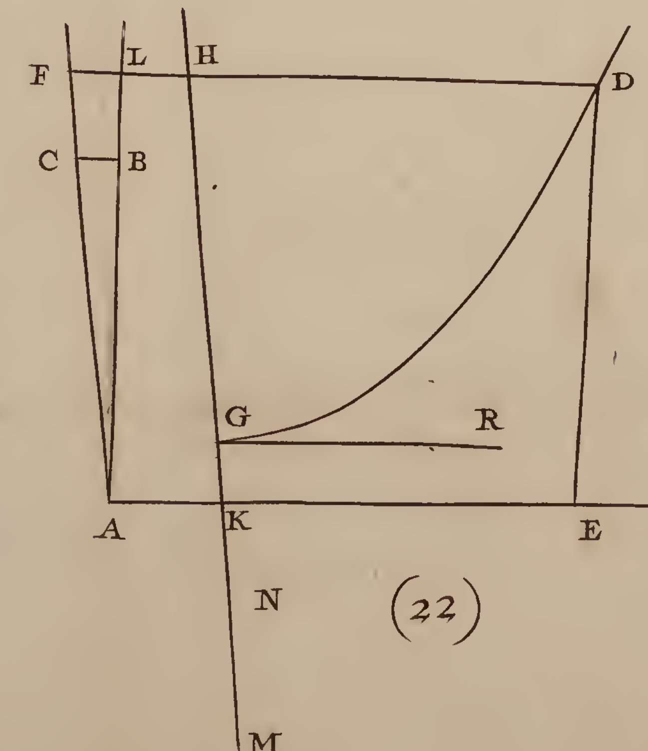

(17)
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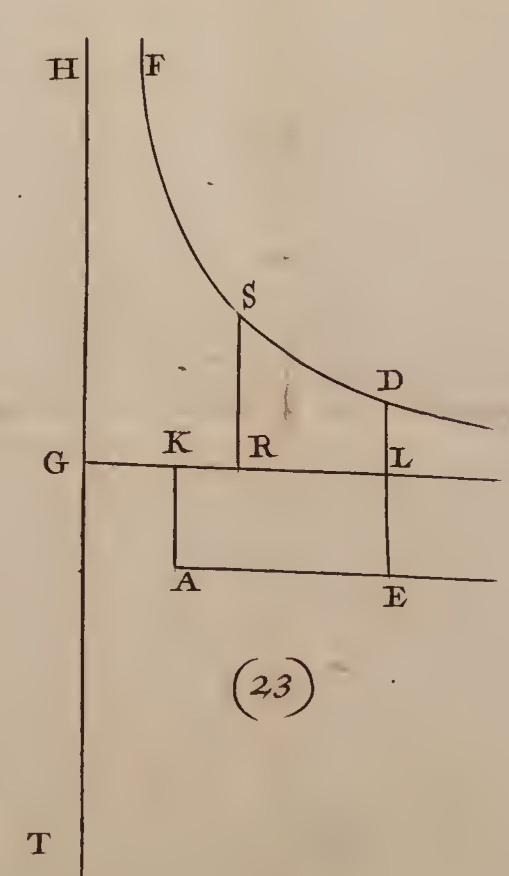

G (18)
H D F L K A E

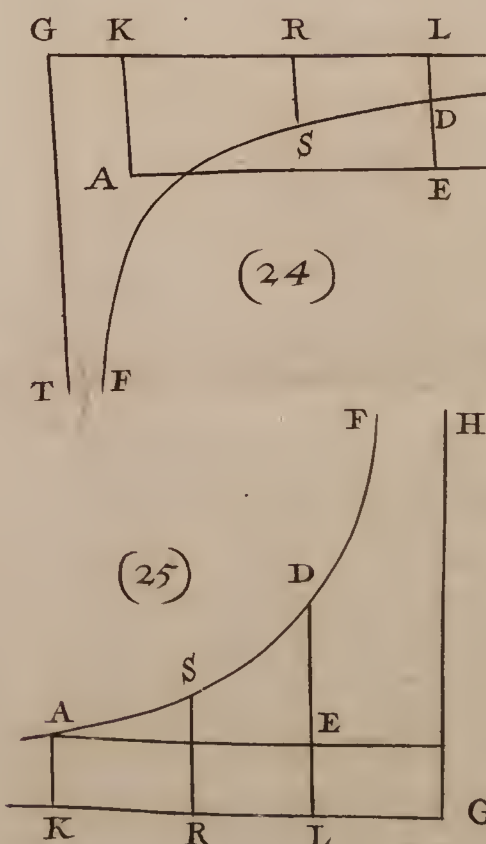

D (19)
H F C K A B E G

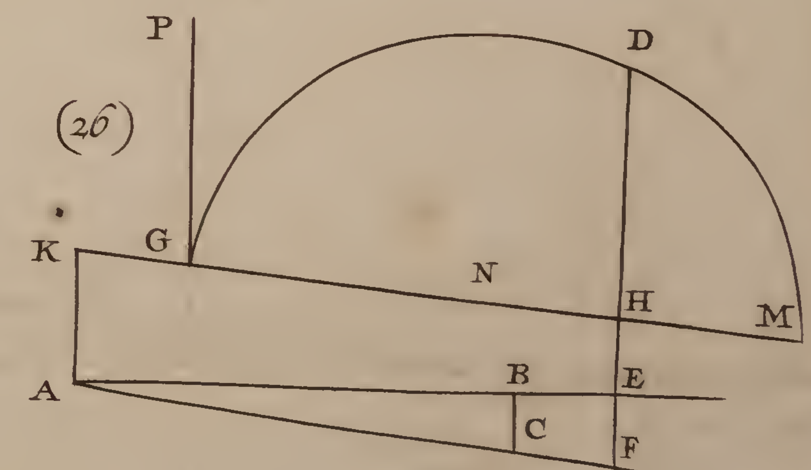

(20)
F L H D C B A K E G

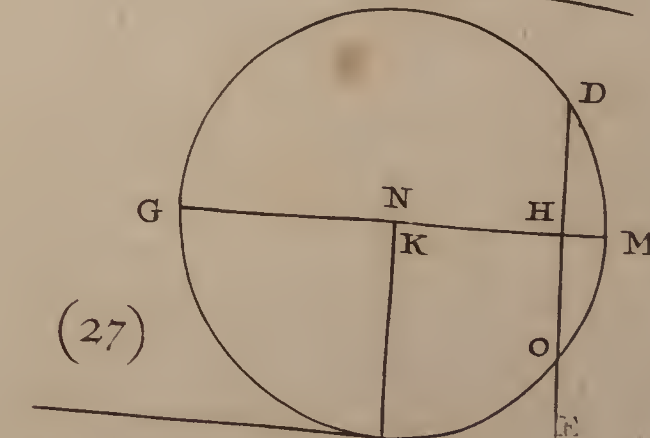

(21)
P D M N K G H A B E C F

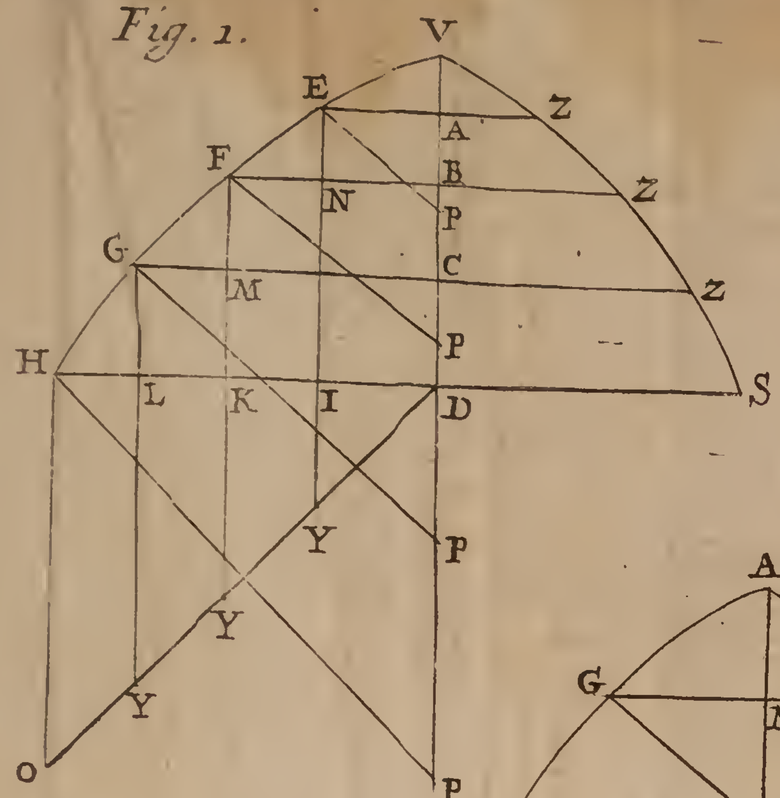

F L H D C B G R A K E N M (22)

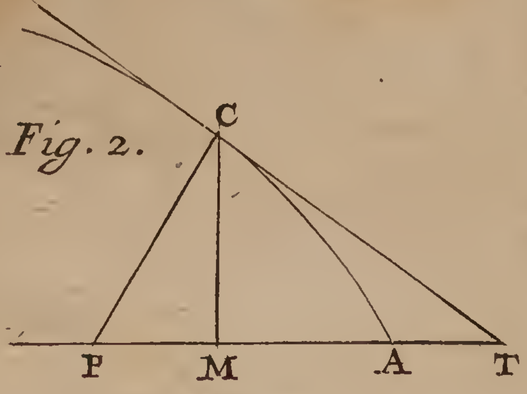

H F S D G K R L A E T (23)

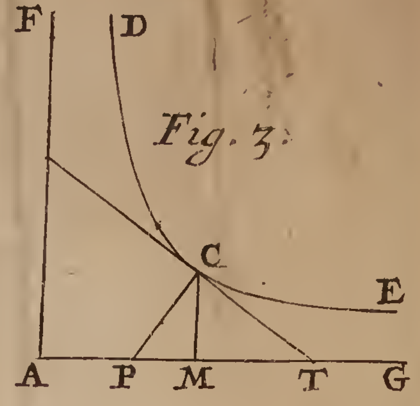

G K R L S D A E (24) T F
F H (25) D S A E K R L G

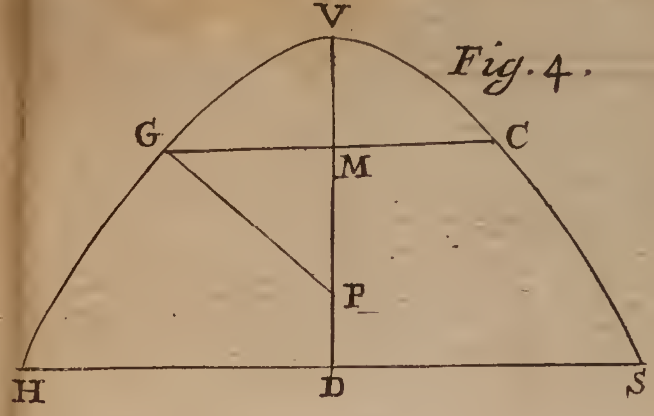

P D (26) G K N H M A B E C F

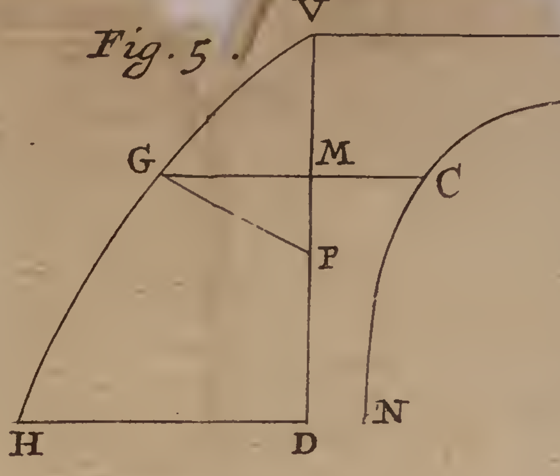

D G N K H M (27) O E

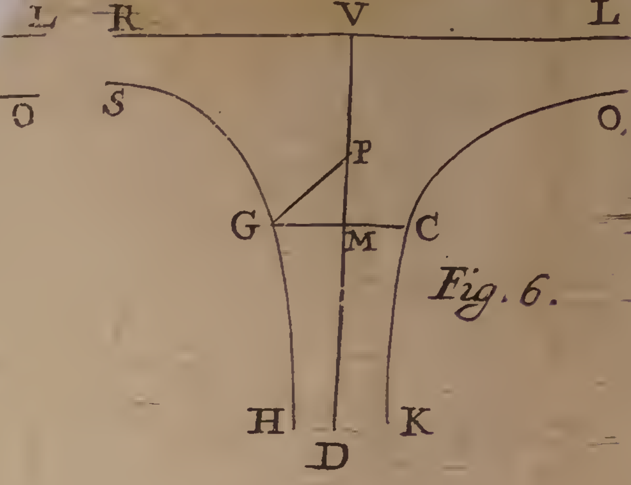

Fig. 1.

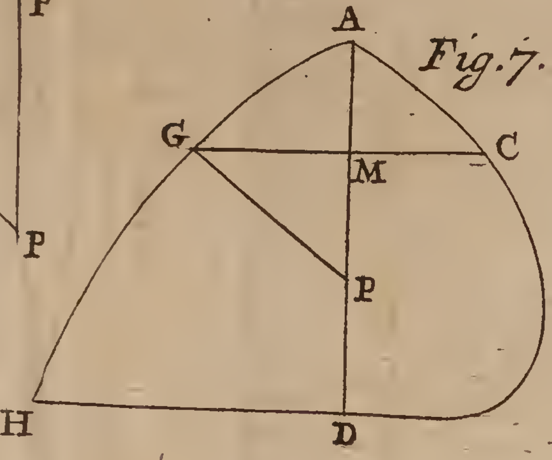

Fig. 2.

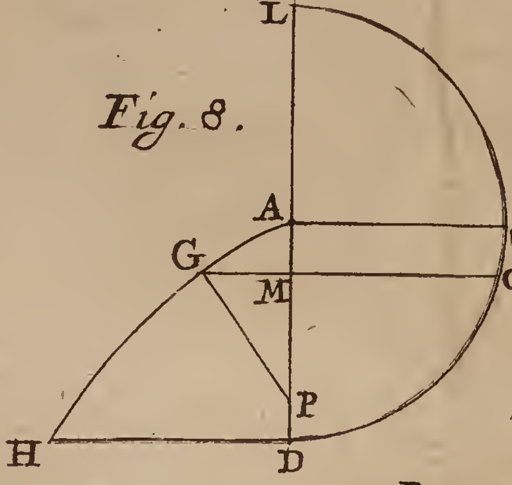

Fig. 3.

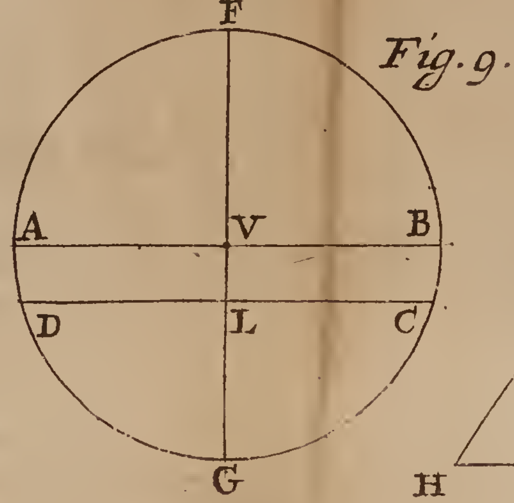

Fig. 4.

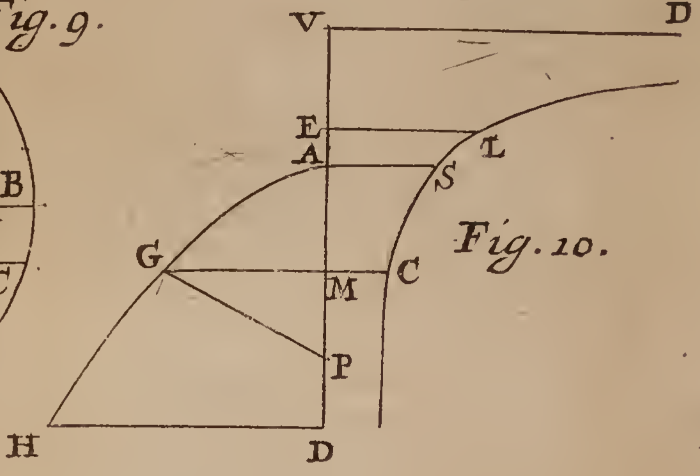

Fig. 5.

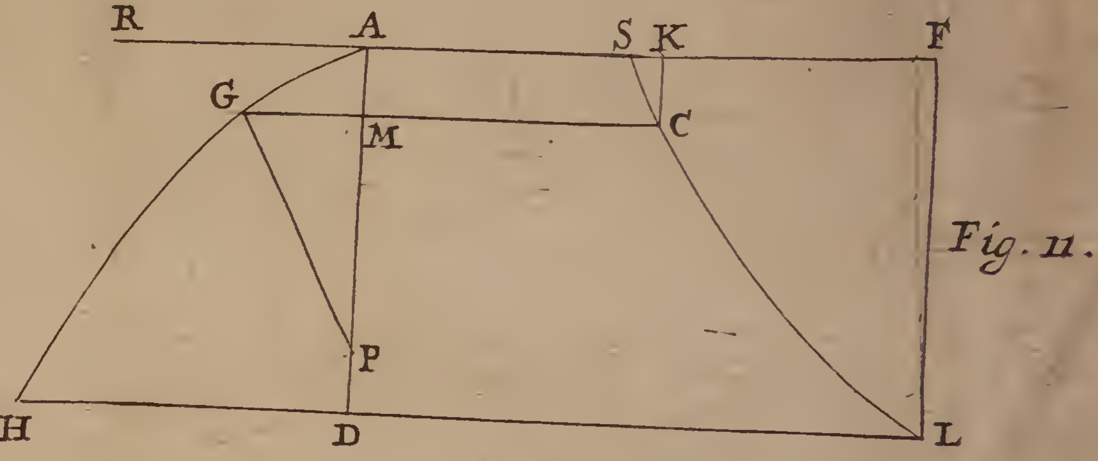

Fig. 6.

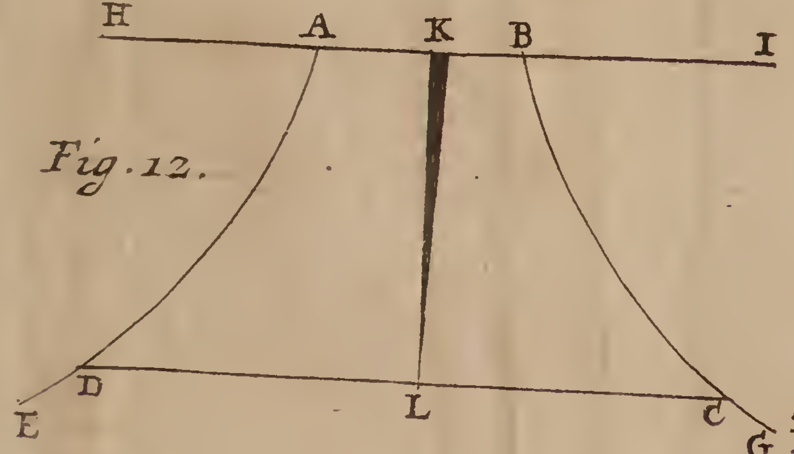

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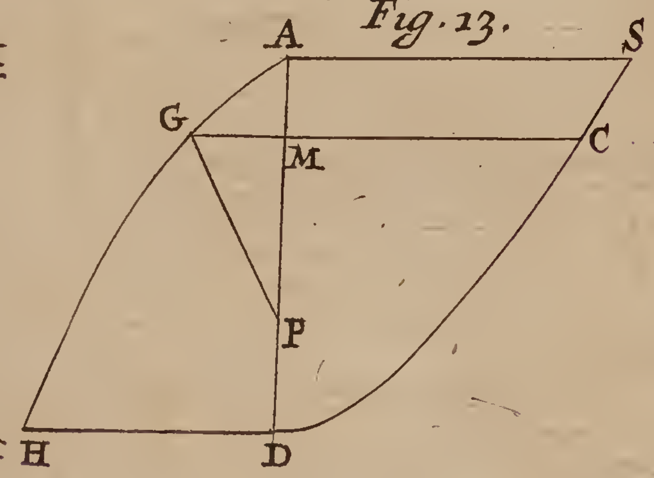

Fig. 8.

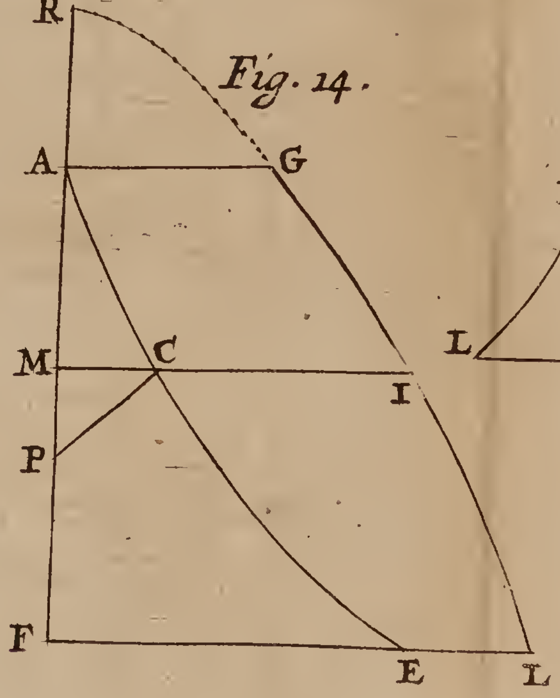

Fig. 9.

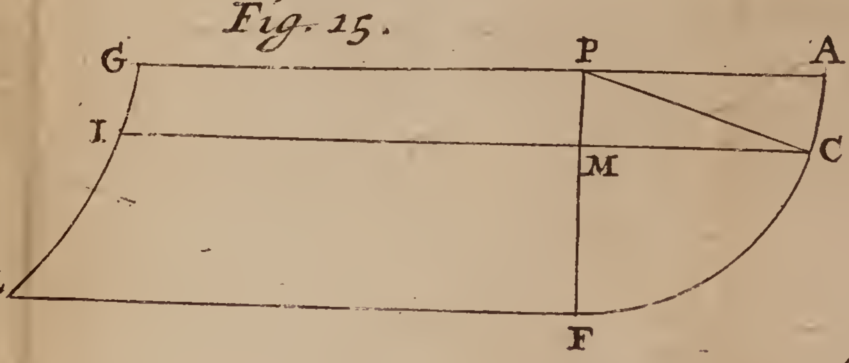

Fig. 10.

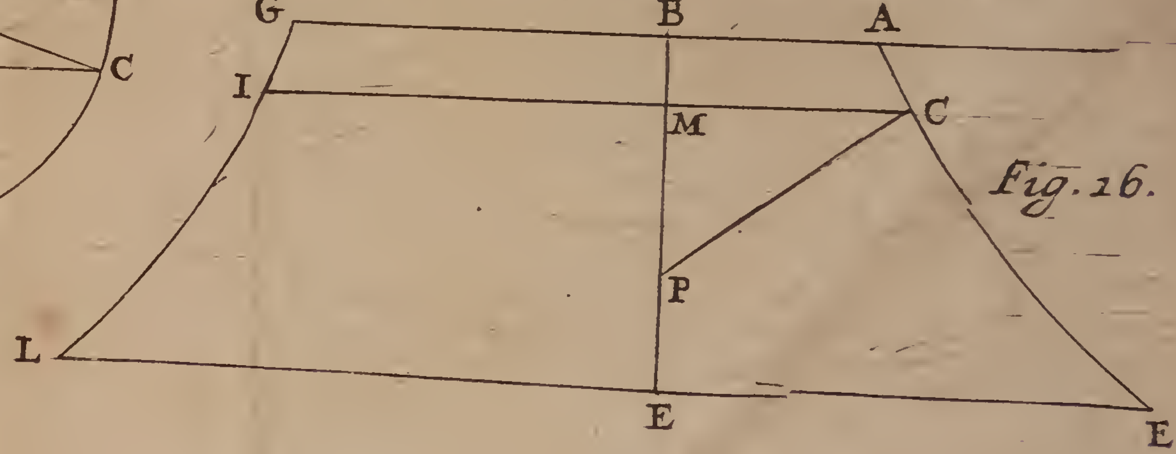

Fig. 11.

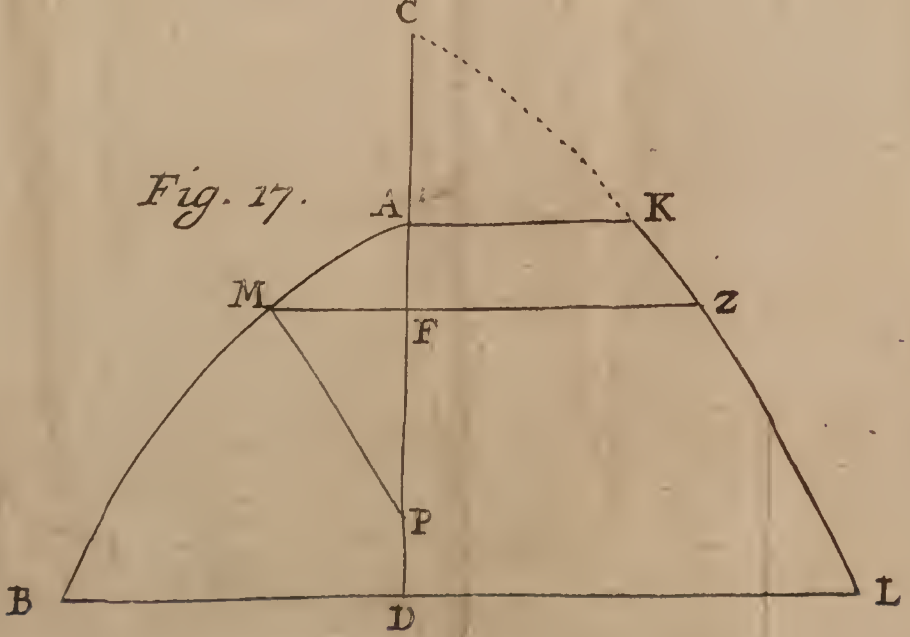

Fig. 12.

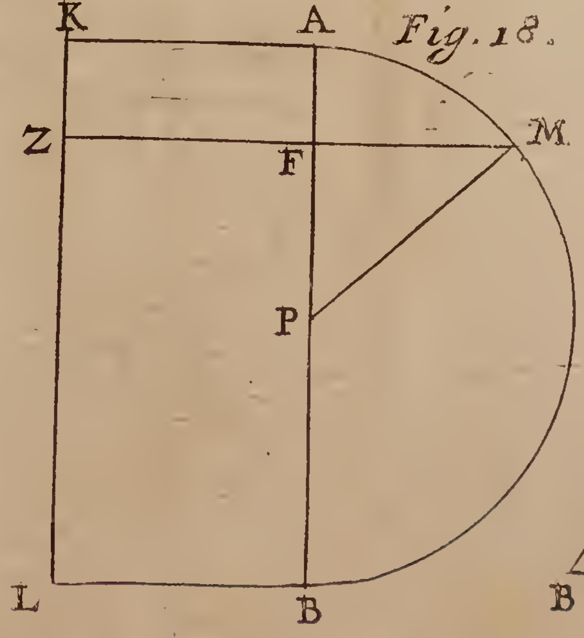

Fig. 13.

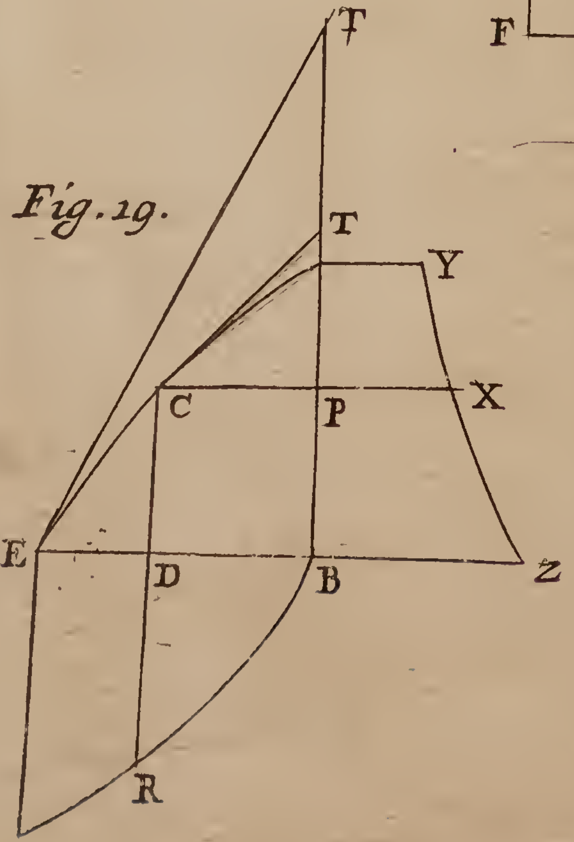

Fig. 14.

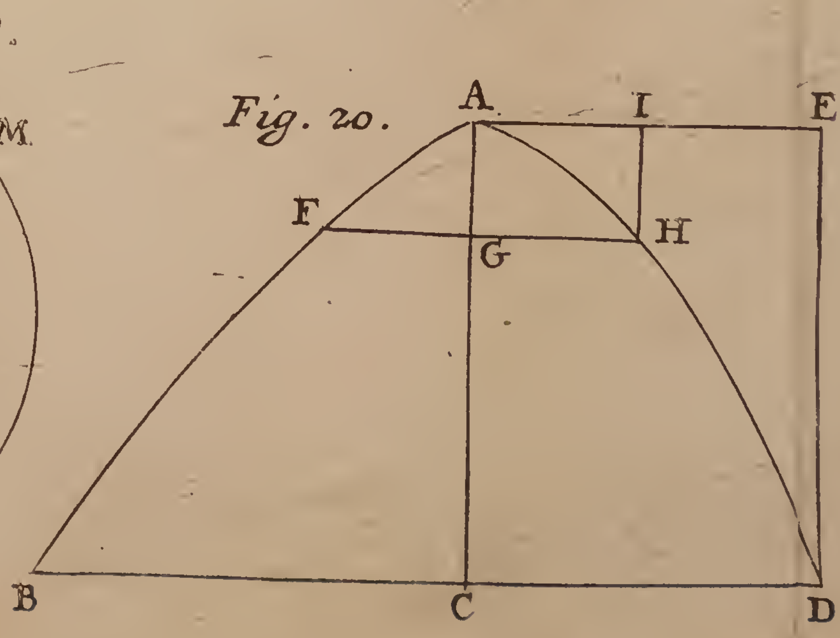

Fig. 15.

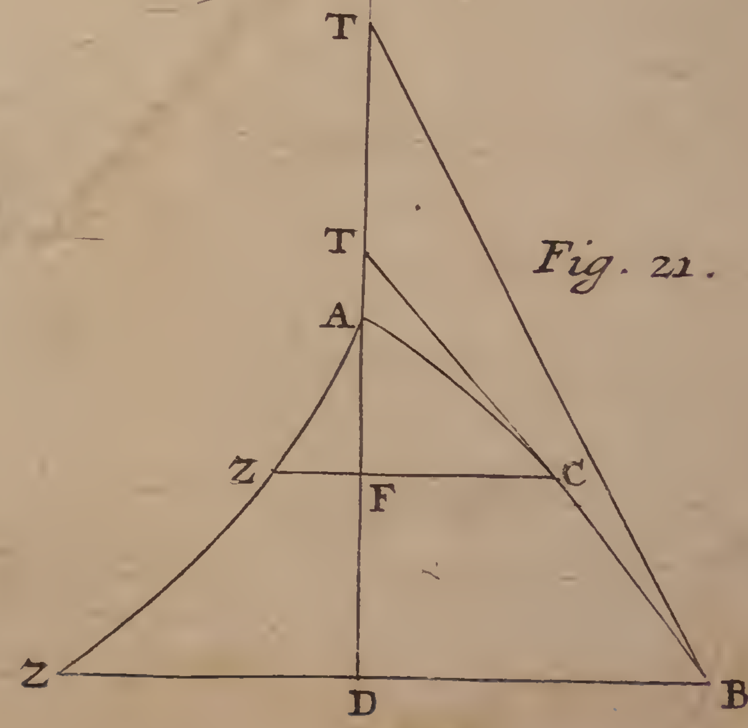

Fig. 16.


Fig. 17.


Fig. 18.


Fig. 19.


Fig. 20.


Fig. 21.

DE FIGURARUM QUADRATURIS.

Pars Prior.

IN Actis Philoſophicis Regiæ Societatis Anni 1686. Specimen exhibui Methodi generalis determinandi Figurarum Quadraturas: cumque poſtea plus otii nactus fueram, credebam me non poſſe illud melius, quam in eadem materia ulterius perficiendâ, collocare; plurima enim tum deerant, quæque me jam feliciter obtinuiſſe ſpero. Ne autem nimium mihi adſcribere, vel aliis detrahere videar, libenter agnoſco Leibnitii Calculum differentialem, tanta mihi in his inveniendis ſuppeditaſſe auxilia, ut ſine illo hæc vix aſſequi potuiſſem eâ, qua optabam facilitate: quantopere ſolidam & ſublimiorem Geometriam hoc uno nobiliſſimo invento adauxerit Celeberrimus ejus Autor, peritiſſimos hujus ævi Geometras latere non poteſt; & quam inſignis fuerit utilitatis, in dimenſionibus Figurarum inveniendis ſequens hic Tractatus ſufficienter indicabit. Abſoluta parte hujus priori, quæ Figuras ſpectat Algebraicas, & quarum Quadratices ſunt etiam Curvæ Algebraicæ; eandem ego Methodum promovere volui ad cæteras Figuras Algebraicas, quarum Areæ non niſi per Curvas tranſcendentes determinari poſſunt. Sed deficiente hìc Calculo Leibnitii differentiali, nova mihi Tangentium Methodus

excogitanda erat, quamque ex principiis tam generalibus deduxi, ut nullam reſpuat tranſcendentis ſpeciem, vel maximè compoſitam. Atque hujus ope Circuli & Hyperbolæ Quadraturæ Tranſcendentes, pari facilitate, qua aliarum Figurarum Quadraturæ Algebraicæ inveniuntur. In eo tamen præſertim nitet non contemnenda Methodi noſtræ præſtantia, quod uno calculo infinitarum Figurarum Quadraturas abſolvat: Et quia infinitæ ſunt Figuræ, quarum Areæ cum ſimplicioribus comparari poſſunt, oſtendam quo pacto comparanda ſit quælibet Figura data cum ſimpliciſſima ejuſdem generis Figura: Unde Theoremata generalia deduco, quibus Quadraturæ particulares abſque omni computationis moleſtia inveniuntur.

Lemma I.

Fig. 1. *Sint duæ Curvæ* FGH, ACS *ita inter ſe relatæ, ut ductâ* PG *perpendiculari ad quodvis Curvæ punctum* G, *ſit intercepta* PM *æqualis lineæ* MC, *quæ eſt ordinatim applicata alterius Curvæ* ACS *ad Axem communem* AD; *erit dimidium Quadrati ordinatæ* GM *in Curva* FGH, *æqualis Areæ Curvilineæ* AMC, *rectis* AM, MC *& curva altera* AC *comprehenſæ, id eſt* $\frac{1}{2}$GMq=AMC. *Demonſtratur hoc Theorema in Lectionibus Geometricis D. D.* Barrow.

Corol. INvenire Quadraturam Areæ cujuſvis Curvilineæ AMC, idem eſt, atque aliam Curvam FGH invenire, cujus intercepta PM ſit æqualis ordinatim applicatæ MC in Figura Quadranda AMC. Cujus pulcherrimi Problematis ſolutionem dabo facilem & generalem, quoniam ex hoc tota noſtra Methodus dependet.

Prob. I.

Datâ expreſſione Analytica interceptæ PM, *æquationem invenire, quæ Curvæ iſtius* FGH *naturam definiat.*

SIT communis utriuſque Curvæ abſciſſa AM=y, ordinatim applicata GM=x, & MC=z, & (a) quantitas data & determinata, Unitatis Locum ſupplens, ad efficiendos (ſi opus fuerit) omnes terminos æqualium dimenſionum. *Solutio*: Reducatur expreſſio Analytica interceptæ PM (ſeu MC) ad formam ſimpliciſſimam, liberando terminum y, quantum fieri poteſt, à ſignis

Radicalibus,

Radicalibus, (ſi talia occurrant) ita tamen ut quantitas y extra vinculum generale non habeat Exponentes diverſæ Denominationis, ab exponentibus ejuſdem, qui ſunt ſub vinculo: Expreſſio ſic reducta multiplicetur per quantitatem y: Et apponantur omnes Poteſtates (quantitatis y) quæ ſub maxima producti poteſtate continentur; tales autem, ut Poteſtates appoſitæ habeant exponentes ejuſdem Naturæ & Denominationis cum exponente maximæ Poteſtatis in producto: Afficiantur hi termini coefficientibus b, c, d, e, f, &c. quantitates incognitas Denotantibus; erunt hi termini (quibuſvis ſignis connexi) altera pars æquationis quæſitæ, cujus altera eſt $xx = GMq$, vel ſaltem quæſitam eminenter continebit. Atque hoc faciendum eſt, ſive exponens maximæ poteſtatis ſit affirmativus, ſeu negativus, integer, ſeu fractus: Ut ſi valor lineæ PM (ſeu MC) in y multiplicatus habeat y^6 maximam Dignitatem quantitatis y extra vinculum, apponendæ ſunt omnes Dignitates, quarum Exponentes continentur ſub 6; erunt itaque termini apponendi y^6, y^5, y^4, y^3, y^2, y^1, y^0 $(=1)$: Vel ſi reperiatur in producto y^{-6} extra vinculum, erunt termini apponendi, y^{-6}, y^{-5}, y^{-4}, y^{-3}, y^{-2}, y^{-1}, y^{-0} $(=1)$: Vel ſi $y^{\frac{7}{2}}$ ſit maxima Dignitas in producto extra vinculum, erunt termini apponendi $y^{\frac{7}{2}}$, $y^{\frac{6}{2}}$, $y^{\frac{5}{2}}$, $y^{\frac{4}{2}}$, $y^{\frac{3}{2}}$, $y^{\frac{2}{2}}$, $y^{\frac{1}{2}}$, $y^{\frac{0}{2}}$ $(=1)$: Vel denique ſit maximus producti terminus $y^{-\frac{5}{3}}$, erunt poteſtates apponendæ $y^{-\frac{5}{3}}$, $y^{-\frac{4}{3}}$, $y^{-\frac{3}{3}}$, $y^{-\frac{2}{3}}$, $y^{-\frac{1}{3}}$, $y^{-\frac{0}{3}}$ $(=1)$. Ex æquatione hoc modo conſtituta inveſtigetur valor analyticus interceptæ PM per Leibnitii Methodum in Actis Eruditorum explicatam; & comparetur ejus valor ſic repertus cum valore ejus dato, ſecundum cognitas Comparationum Leges à Carteſio expoſitas, unde novæ æquationes reſultabunt, quæ coefficientes b, c, d, e, &c. determinabunt: Et coefficientium valores in æquatione ſubſtituti æquationem quæſitam præciſe conſtituent; rejectis iis terminis, quorum coefficientes nihilo æquales inveniuntur, vel abſurdum involvunt.

Pag. 467. *An.* 1684.

Schol. Sicubi ordinatim applicata MC, vel intercepta PM ad ſimpliciſſimam formam reducta habeat diverſas poteſtates quantitatis y extra vinculum univerſale, tum apponendæ erunt poteſtates ſub ſingulis contentæ: Vel ſi plura habuerit vincula compoſita, tum quod hic cum uno faciendum præſcribitur, cum reliquis pariter fieri debet. Sequentia Exempla hæc omnia illuſtrabunt.

Exemplum 1. Invenienda ſit Quadratura Figuræ AMC cujus Natura exprimitur hac æquatione $z^2 = y^4 + aayy$ valor ordinatæ z ad ſimpliciſſimam formam reductæ erit $z = y\sqrt{yy+aa}$: Ut habeatur hujus Figuræ Quadratura, invenienda eſt alia Curva FGH in qua intercepta PM ſit $y\sqrt{y^2+a^2}$, ut patet ex Corolario Lemmatis 1; ideo juxta Regulam in Solutione Problematis 1. præſcriptam, multiplicandus eſt valor datus lineæ PM (ſeu MC) per y, unde productum erit $yy\sqrt{y^2+a^2}$: Jam quia maxima dignitas extra vinculum eſt y^2, ideo apponendi ſunt omnes termini inferiores ſcil. y^2, y^1, y^0 $(=1)$ ipſo ſemper maximo termino incluſo, qui coefficientibus incognitis affecti æquari debent Quadrato quantitatis x, unde æquatio quæſitam eminenter continens erit $by^2+cay+ea^2 \times \sqrt{y^2+a^2} = xx$. Ex hac æquatione inveſtigetur valor Analyticus Lineæ PM per Leibnitii Methodum hoc modo; compendii gratia ponatur $by^2+cay+ea^2 = p$, atque $\sqrt{y^2+a^2} = q$, unde $pq = xx$, cujus æquatio differentialis eſt $pdq+qdp = 2xdx$, ſed $dq = \frac{ydy}{\sqrt{yy+aa}}$, & $dp = 2bydy+cady$, reſtituantur itaque valores quantitatum p, q, nec non dp, dq in æquatione differentiali, eritque illa hujuſmodi,

$$\frac{by^3+cay^2+ea^2y \times dy}{\sqrt{y^2+a^2}} + 2by\sqrt{y^2+a^2} \times dy + ca\sqrt{y^2+a^2} \times dy = 2xdx.$$

Et omnibus terminis ſub eodem communi denominatore reductis,

erit $\frac{3by^3+2cay^2+ea^2y+2ba^2y \times dy}{\sqrt{yy+aa}} = 2xdx$

hæc æquatio differentialis in Analogiam reſoluta dabit,

$$dy \,.\, dx :: 2x \,.\, \frac{3by^3+2cay^2+ea^2y+2ba^2y+ca^3}{\sqrt{y^2+a^2}} :: x \,.\, PM.$$ Et proinde

invenietur $PM = \frac{3by^3+2cay^2+ea^2y+2ba^2y+ca^3}{2\sqrt{y^2+a^2}} = y\sqrt{y^2+a^2}$:

Hæc æquatio à fractis & ſurdis liberata, multiplicando totam per denominatorem prioris partis dabit.

$$\left.\begin{matrix} 3by^3+2cay^2+ea^2y+ca^3 \\ +2ba^2y \end{matrix}\right\} = 2y^3+2a^2y$$

Comparando

Comparando itaque terminos hujus æquationis, erit primò $3b=2$, unde $b=\frac{2}{3}$; ſecundò $2c=0$, unde $c=0$; tertiò $e+2b=2$, unde $e=\frac{2}{3}$; quarto denique $c=0$, ut prius: Ex quibus manifeſtum eſt terminum à coefficiente c affectum æquationis quæſitæ compoſitionem non ingredi, ſed ſolos terminos à coefficientibus b, & e affectos, quarum valores in æquatione aſſumpta ſubſtituti dabunt quæſitam ſcil. $\frac{2y^2+2a^2}{3}\times\sqrt{y^2+a^2}=xx$, quæ definit Curvam FGH in qua intercepta PM eſt æqualis ordinatæ MC in Figura quadranda AMC, ideoque per Lemma 1. erit ejus Area $=\frac{y^2+a^2}{3}\times\sqrt{y^2+a^2}=$ $=\frac{1}{2}xx$; quæ non competit portioni AMC, ſed eandem excedit toto ſpatio $\frac{1}{3}a^3$, quod ſuo loco fuſius explicabitur.

Exemp. 2. Eſto natura Curvæ ACS talis $z^2=y^3+ay^2$, & inveniendа ſit ejus Areæ Dimenſio: Ordinata ad ſimpliciſſimam formam reducta eſt $z=y\sqrt{y+a}$. Et juxta præſcriptum Regulæ primi Problematis $by^2+cay+da^2\times\sqrt{y+a}=xx$ eſt æquatio eminenter continens illam, quæ definit Curvam quæſitam FGH, (in qua $PM=y\sqrt{y+a}=MC$) ex qua æquatione inveniatur valor lineæ PM ut jam explicui; hic valor adæquetur valori ejus dato; liberetur hæc æquatio à fractis & ſurdis, & dein termini ritè inter ſe comparentur; ex his comparationibus invenies $b=\frac{4}{5}$, $c=\frac{4}{15}$, $d=-\frac{8}{15}$, hi coefficientium valores in propriis locis ſubſtituti dabunt $\frac{4}{5}y^2+\frac{4}{15}ay-\frac{8}{15}a^2\times\sqrt{y+a}$ $=xx$, quæ definit Curvam quæſitam FGH, cujus intercepta PM eſt $y\sqrt{y+a}$; & proinde per Lemma 1.

$$\text{Area quæſita}=\frac{6y^2+2ay-4aa}{15}\times\sqrt{y+a}=\frac{1}{2}xx.$$

Exemp. 3. Determinanda ſit Quadratura Figuræ AMC cujus proprietas eſt hujuſmodi $z=\frac{1}{y^3}\sqrt{\frac{ay^8-a^9}{y}}$; hic valor lineæ PM in y multiplicatus eſt $\frac{1}{y^2}\sqrt{\frac{a^8y-a^9}{y}}$; ubi maxima dignitas extra vinculum eſt $\frac{1}{y^2}=y^{-2}$; appoſitis itaque poteſtatibus inferioribus cum coefficientibus

cientibus incognitis erit $\frac{b}{yy} + \frac{c}{ay} + \frac{d}{aa}\sqrt{\frac{a^8y - a^9}{y}} = xx$, ex qua invenio valorem interceptæ PM, quem comparo cum valore ejus dato, & sic innotescunt coefficientes b, c, d, scil. $b = \frac{4}{5}$, $c = -\frac{4}{15}$, $d = -\frac{8}{15}$, quibus in æquatione substitutis erit $\frac{4}{5yy} - \frac{4}{15ay} - \frac{8}{15aa} \times \sqrt{\frac{a^8y - a^9}{y}} = x^2$; quæ definit Curvam FGH, in qua PM=MC. Ergo per Lemma 1.

$$\text{Area quæsita} = \frac{2}{5yy} - \frac{2}{15ay} - \frac{4}{15aa} \times \sqrt{\frac{a^8y - a^9}{y}} = \tfrac{1}{2}xx.$$

Exemp. 4. Sit $z = \sqrt{y} \times \sqrt{a + \sqrt{y^3}}$ proprietas Figuræ AMC, cujus Area est inveniend: ut hæc habeatur, invenienda primò est alia Curva FGH in qua PM $= \sqrt{y} \times \sqrt{a + \sqrt{y^3}}$ seu $y^{\frac{1}{2}} \times \sqrt{a + y^{\frac{3}{2}}}$: Jam verò hic valor lineæ PM in y multiplicatus dat $y^{\frac{3}{2}} \times \sqrt{a + y^{\frac{3}{2}}}$, & maxima dignitas extra vinculum compositum est numerus fractus, appositis itaque potestatibus inferioribus cum coefficientibus incognitis, erit $by^{\frac{3}{2}} + cy^{\frac{2}{2}} + dy^{\frac{1}{2}} + ey^{\frac{0}{2}} \times \sqrt{a + y^{\frac{3}{2}}} = x^2$, hoc est
$b\sqrt{y^3} + c\sqrt{ay^2} + d\sqrt{a^2y} + e\sqrt{a^3} \times \sqrt{a + \sqrt{y^3}} = xx$; ex hac æquatione inveniatur valor lineæ PM, qui cum valore ejus dato comparatus determinabit coefficientes, scil. $b = \frac{8}{9}$, $c = 0$, $d = 0$, $e = \frac{8}{9}$, unde
$\frac{8\sqrt{y^3} + 8\sqrt{a^3}}{9} \times \sqrt{a + \sqrt{y^3}} = xx$, quæ æquatio definit Curvam FGH in qua intercepta PM æqualis est ordinatim applicatæ datæ Figuræ AMC. Ergo per Lemma 1.

$$\text{Area quæsita} = \frac{4\sqrt{y^3} + 4\sqrt{a^3}}{9} \times \sqrt{a + \sqrt{y^3}} = \tfrac{1}{2}xx.$$

Exemp. 5. Quæritur quadratura Figuræ AMC, quam comprehendit Curva AC cujus æquatio est $z^2 = \frac{y^2}{y + a}$, unde valor ordinatæ $z = y \times \frac{1}{\sqrt{y + a}}$, vel secundum aliam notandi formulam $z = y\overset{-2}{\sqrt{y + a}}$: vel tertio $z = y \times \overline{y + a}|^{-\frac{1}{2}}$, quoniam hic casus nonnullam habet difficultatem, visum est Leibnitii Calculum, juxta omnes illas notandi formulas illustrare. Ex prima itaque erit

$byy+$

$\frac{byy + cay + ea^2}{\sqrt{y+a}} = x^2$, compendii gratia ponatur $byy + cay + ea^2 = = p$, $\sqrt{y+a} = q$, unde æquatio erit $\frac{p}{q} = xx$, cujus æquatio differentialis ex Autoris regula Divisionis pro priori, & potentiarum pro secunda parte dabit $\frac{\pm pdq \mp qdp}{qq} = 2xdx$. Signa ambigua quod attinet, notandum est, illa duobus modis posse explicari, quamvis Curva FGH sit adhuc incognita, & primo quidem ex data Curva ACS innotescet an crescentibus abscissis, crescant pariter ordinatæ; an contra: (quoniam Area Curvæ datæ æqualis est dimidio Quadrati ordinatæ in Curva incognita FGH) ideoque secundum ea, quæ Autor habet de signis ambiguis Pag. 468. *Act. Erud.* Anni 1684. eorum etiam valor innotescet. Secundo explicari possunt signa illa ambigua per comparationem factam cum terminis utriusque valoris interceptæ PM. Jam quia crescente abscissa AM, crescit pariter Area AMC, ideoque etiam incognita ordinata GM, concludendum est fractionem in præcedenti æquatione differentialem ita explicandam esse, ut ejus valor sit affirmativus, quod fiet si prius signum statuatur negativum, posterius verò affirmativum, hoc est $\frac{-pdq + qdp}{qq} = 2xdx$. Et restitutis valoribus quantitatum p, & q erit

$$\left.\frac{\frac{-by^2 - cay - eaa \times dy}{2\sqrt{y+a}} + \sqrt{a+y} \times 2bydy + cady}{y+a}\right\} = 2xdx$$

Quæ in Analogiam resoluta dabit incognitum valorem interceptæ PM, nimirum,

$$\left.\frac{\frac{-by^2 - cay - eaa}{2\sqrt{y+a}} + \sqrt{y+a} \times 2by + ca}{2y+2a}\right\} = \frac{y}{\sqrt{y+a}} = z = \text{PM}.$$

Atque hæc æquatio à fractis & surdis liberata (quæ operatio semper facillime perficitur) dabit

$$\left.\begin{matrix} 3by^2 + cay + 2ca \\ + 4bay - eaaa \end{matrix}\right\} = 4y^2 + 4ay$$

Erit

Erit prima comparatio $3b = 4$, unde $b = \frac{4}{3}$; ſecunda $\bar{c} + 4b = 4$, unde $c = -\frac{4}{33}$ tertia $2c - e = 0$, unde $e = -\frac{8}{3}$. Et proinde æquatio Curvam FGH definiens eſt

$$\frac{4y^2 - 4ay - 8aa}{3\sqrt{y+a}} = x^2; \text{ \& Area } = \frac{2y^2 - 2ay - 4aa}{3\sqrt{y+a}} = \frac{1}{2}xx =$$
$$= \frac{2}{3}\sqrt{y^3 - 3ay^2 + 4a^3}$$

Ex ſecundo modo deſignandi ordinatam z (ſeu lineæ PM valorem) erit juxta Problematis primi ſolutionem

$\overline{byy + cay + eaa} \times \sqrt[-2]{y+a} = xx$; compendii gratia $byy + cay + eaa = p$, $\sqrt[-2]{y+a} = q$, unde $pq = xx$; cujus æquatio differentialis ex Autoris regula multiplicationis pro priori, & potentiarum pro poſteriori parte dabit $pdq + qdp = 2xdx$. Atqui per Regulam radicum $dq = \frac{dy}{-2\sqrt[-2]{y+a}}$, & potentiarum $dp = \overline{2by + ca} \times dy$; quæ in æquatione differentiali ſubſtitutæ dabunt

$$\frac{\overline{by^2 + cay + eaa} \times dy}{-2\sqrt[-2]{y+a}} + dy\sqrt[-2]{y+a} \times \overline{2by + ca} = 2xdx$$

Ex qua æquatione in Analogiam reſoluta invenietur expreſſio Analytica, lineæ PM, quæ æquata expreſſioni ejuſdem datæ erit

$$\left.\frac{-3by^2 - cay - 4bay + eaa - 2caa}{-4\sqrt[-2]{y+a}}\right\} = y\sqrt[-2]{y+a}$$

Ex hac æquatione à fractis & ſurdis liberata reſultabit demum hæc æquatio.

$$\left.\begin{array}{l} -3byy - cay + eaa \\ -4bay - 2caa \end{array}\right\} = -4yy - 4ay.$$

Erit prima comparatio $-3b = -4$, unde $b = \frac{4}{3}$; ſecunda $-c - 4b = -4$, unde $c = -\frac{4}{3}$; tertia denique $e - 2c = 0$, unde $e = -\frac{8}{3}$: Ipſi nimirum valores, qui in priori calculo inventi ſunt: Et proinde

$$\frac{4yy - 4ay - 8aa}{3} \times \sqrt[-2]{y+a} = xx; \text{ Area} = \frac{yy - ay - 2aa}{3} \times 2\sqrt[-2]{y+a} =$$
$$= \frac{2}{3}\sqrt{y^3 - 3ayy + 4a^3} = \frac{1}{2}xx.$$

Ex

Ex tertia itaque notandi formula erit $byy + cay + eaa \times \overline{y + a}|^{-\frac{1}{2}} = xx$; Cujus æquatio Differentialis per Leibenitii regulas eſt

$$byy + cay + eaa \times -\tfrac{1}{2}\overline{y + a}|^{-\frac{3}{2}} + \overline{y + a}|^{-\frac{1}{2}} \times 2by + ca \Big\}\, dy = 2xdx.$$

ex qua invenietur valor lineæ PM, qui valori ejus dato æquatus, & ablatis ſurdis & fractis, $\left.\begin{matrix} \frac{3}{2}by^2 + \frac{1}{2}cay - \frac{1}{2}eaa \\ +2b \quad + caa \end{matrix}\right\} = 2y^2 + 2ay$: Unde prima comparatio $\frac{3b}{2} = 2$, unde $b = \frac{4}{3}$; ſecunda $\frac{1}{2}c + 2b = 2$, unde $c = -\frac{4}{3}$; tertia $-\frac{1}{2}e + c = o$, unde $e = -\frac{8}{3}$. Reſtitutis valoribus coefficientium modo inventis, erit

$$\tfrac{4}{3} \times yy - ay - 2a^2 \times \overline{y + a}|^{-\frac{1}{2}} = x^2;\ \text{Area} = \tfrac{2}{3} \times yy - ay - 2aa \times \overline{y + a}|^{-\frac{1}{2}} = \tfrac{2}{3}\sqrt{y^3 - 3ay^2 + 4aa} = \tfrac{1}{2}xx.$$

Exemp: 6. Sit $z^2 a^5 = 9y^7 + 21ay^6 + 16aay^5 + 4a^3y^4$: Simpliciſſima Ordinatæ z expreſſio analytica eſt $z = 3y^2 + 2ay \times \sqrt{\frac{y^3 + ay^2}{a^5}}$: Unde juxta ſolutionem primi Problematis

$$by^3 + cay^2 + daay + ea^3 \times \sqrt{\frac{y^3 + ay^2}{a^5}} = xx.$$

Et determinatis coefficientibus ut jam oſtenſum eſt, invenietur $b = \frac{4}{3}$, $c = \frac{4}{3}$, $d = o$, $e = o$; Ex quibus manifeſtum eſt

$$\text{Aream} = \frac{2y^3 + 2ay^2}{3aa} \times \sqrt{\frac{y^3 + ay^2}{a}} = \tfrac{1}{2}xx.$$

Atque hoc modo habentur cæterarum Figurarum Quadraturæ, quando per æquationem finitam explicari poſſunt; & exempla jam adducta ſatis oſtendunt, qualis in ſimilibus fieri debeat proceſſus; nihilque difficultatis hic occurrere poteſt, dummodo Lector in ſæpe memorato Leibnitii Calculo verſatus fuerit. Pars vero Methodi noſtræ longe præſtantior hæc eſt, qua uno hujuſmodi calculo infinitarum Figurarum Quadraturæ determinantur, quarum ne una quidem, ante ſpecimen noſtrum in Actis Philoſophicis

editum,

editum, inveniri poſſit certa Methodo publici juris factâ.

Exemp. 7. Inveniendæ ſint Quadraturæ omnium Figurarum AMC, quarum naturæ hac æquatione generali $z = y\sqrt[r]{y+a}$ definiuntur, in qua r denotat exponentem vinculi radicalis Quadratici, Cubici, *&c.* Ex problemate primo conſtat $by^2 + cay + eaa \times \sqrt[r]{y+a} = xx$ eminenter continere Curvam FGH in qua PM = = MC; ut ergo habeatur valor Analyticus interceptæ PM, ponatur compendii gratia $by^2 + cay + eaa = p$, $\sqrt[r]{y+a} = q$ unde $pq = xx$, cujus æquatio differentialis eſt $pdq + qdp = 2\,xdx$; ſed per Leibnitii Regulas $dq = \frac{dy}{r\sqrt[r]{\overline{y+a}|^{r-1}}}$, $dp = 2\,bydy + cady$,

quare $$\frac{by^2 + cay + eaa \times dy}{r\sqrt[r]{\overline{y+a}|^{r-1}}} + dy\sqrt[r]{y+a} \times 2\,by + ca = 2\,xdx.$$

Quæ in Analogiam reſoluta, & rite tractata, ut in præcedentibus, dabit,

$$\left.\frac{byy + 2\,rby^2 + cay + 2\,rbay + rcay + eaa + rcaa}{2\,r\sqrt[r]{\overline{y+a}|^{r-1}}}\right\} =$$

$$= y\sqrt[r]{y+a} = z = \text{PM}$$

hæc æquatio à fractis & ſurdis liberata, multiplicando per denominatorem partis, erit

$$\left.\begin{matrix} byy + cay + eaa \\ rbyy + 2\,rbay + rcaa \\ + rcay \end{matrix}\right\} = 2\,ryy + 2\,ray.$$

Erit

Erit prima comparatio $b + 2rb = 2r$, unde $b = \frac{2r}{2r+1}$;

Secunda comparatio $c + 2rb + rc$, unde $c = \frac{2r}{\overline{2r+1} \times \overline{r+1}}$;

Tertia comparatio $e + rc = o$, unde $e = -rc = -\frac{2rr}{\overline{2r+1} \times \overline{r+1}}$;

quibus in æquatione propria ſubſtitutis, erit,

$$\frac{2ryy}{2r+1} + \frac{2ray - 2rraa}{\overline{2r+1} \times \overline{r+1}} \times \sqrt[r]{y+a} = xx$$

Ideoque per Lemma præcedens manifeſtum eſt fore etiam

$$\text{Aream} = \frac{ryy}{2r+1} + \frac{ray - rraa}{\overline{2r+1} \times \overline{r+1}} \times \sqrt[r]{y+a} = \tfrac{1}{2}xx$$

Quæ exprimit Quadraturas omnium Figurarum, quarum Naturæ definiuntur prædicta æquatione generali; & Quadratura cujuſvis figuræ particularis ſub hac incluſæ habetur ſubſtituendo particularem exponentis r valorem in hac Quadratura generali: Sic ſi $r = 2$, habetur Quadratura particularis Secundi Exempli; vel ſi $r = -2$; habetur Area Figuræ quinto Exemplo propoſitæ; ut de aliis Infinitis nihil dicam, quæ eadem facilitate inde eliciuntur.

Exemp. 8. Inveniendæ ſint Quadraturæ omnium Figurarum, AMC, quarum curvæ definiuntur hac generali æquatione $z = yy\sqrt[r]{y+a}$: per Problemata noſtrum primum erit $\overline{by^3 + cayy + faay + eaa} \times \sqrt[r]{y+a} = xx$, ex qua inventus valor Lineæ PM æquetur dato ejus valori; & æquatio à fractis & ſurdis liberata erit

$$\left.\begin{array}{l} byyy + cayy + faay + eaaa \\ 3rbyyy + 3rb + 2rc + rf \\ \quad + 2rc + rf \end{array}\right\} = 2ryyy + 2rayy.$$

Facta comparatione terminorum ultimæ æquationis, erit $b + 3rb = 2r$, unde $b = \frac{2r}{3r+1}$;

Secundò, $c + 3rb + 2rc = 2r$, unde $c = \frac{2r}{\overline{3r+1} \times \overline{2r+1}}$

Tertiò, $f + 2rc + rf = 0$, unde $f = -\frac{4rrr}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1}}$

Quarta denique comparatio $e + rf = 0$, unde $e = \frac{4rrr}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1}}$;

Substituantur hi valores coefficientium in propria æquatione, & invenietur.

$$\text{Area} = \frac{ry^3}{3r+1} + \frac{rayy}{\overline{3r+1} \times \overline{2r+1}} - \frac{2rraay + 2rrra^3}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1}} \times$$
$$\times \sqrt[r]{y+a} = \tfrac{1}{2}xx.$$

Cum ex duobus ultimis Exemplis observassem coefficientium valores regulari quodam ordine progredi, suspicatus sum eundem progressum in infinitum usque continuari; placet enim *Naturæ* constans & perpetua Uniformitas. Et facto periculo, rem ita se habere compertum habui. Processus erat ejusmodi.

Sit $z = y^n \times \sqrt[r]{y+a}$ æquatio à qua definiuntur omnes curvæ ACS, quæ comprehendunt totidem Areas AMC, quarum Quadraturæ sunt Inveniendæ. Juxta Solutionem primi Problematis, multiplicetur hic valor Ordinatæ z (seu interceptæ PM) per y; eritque productum $y^{n+1} \times \sqrt[r]{y+a}$: Ubi indefinitus numerus $n+1$ est exponens maximæ dignitatis extra vinculum; ideoque potestates inferiores adjiciendæ sunt infinitæ, nimirum juxta Regulam nostram.

$$by^{n+1} + cy^{n} + dy^{n-1} + ey^{n-2} + fy^{n-3} \ \&c. \times \sqrt[r]{y+a} = xx.$$

Ex

Ex hac æquatione investigo valorem Lineæ PM per Leibnitii Methodum, qui æquatus valori ejus dato erit post usitatam reductionem,

$$\left.\begin{array}{lllll} \overline{n+1}\times rby^{n+1} & + nrcy^{n} & + \overline{n-1}\times rdy^{n-1} & + \overline{n-2}\times rey^{n-2} & + \overline{n-3}\times rfy^{n-3} \\ +b & + \overline{n+1}\times rb & + n\times rc & + \overline{n-1}\times rd & + \overline{n-2}\times re \;\&c. \\ & +c & +d & +e & +f \end{array}\right\} = 2ry^{n+1} + 2ray^{n}.$$

Facta jam comparatione omnium terminorum hujus æquationis, invenientur coefficientium Determinationes hujusmodi

Prima, $b = \dfrac{2r}{\overline{n+1}\times r+1}$

Secnnda, $c = \dfrac{2r}{\overline{\overline{n+1}\times r+1}\times\overline{n\times r+1}}$

Tertia, $d = -\dfrac{n\times 2r^{2}}{\overline{\overline{n+1}\times r+1}\times\overline{n\times r+1}\times\overline{\overline{n-1}\times r+1}}$

Quarta, $e = \dfrac{n\times\overline{n-1}\times 2r^{3}}{\overline{\overline{n+1}\times r+1}\times\overline{n\times r+1}\times\overline{\overline{n-1}\times r+1}\times\overline{\overline{n-2}\times r+1}}$

Quinta, $f = -\dfrac{n\times\overline{n-1}\times\overline{n-2}\times 2r^{4}}{\overline{\overline{n+1}\times r+1}\times\overline{n\times r+1}\times\overline{\overline{n-1}\times r+1}\times\overline{\overline{n-2}\times r}\times\overline{\overline{n-3}\times r+1}}.$

Ex compositione quinque harum coefficientium liquet, quomodo cæteræ omnes in infinitum absque ulteriori calculo formari possunt: præterea ob progressionem $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}$ &c. in numeratore coefficientium, si (n) sit numerus integer & positivus, vel nihilo æqualis, tum Figuræ AMC Quadratrix FGH erit Curva Algebraica, in cujus æquatione tot semper erunt coefficientes b, c, d, &c. quot sunt Unitates in $n+1$; & eorum signa quod attinet, post duo priora, quæ semper sunt affirmativa; negativum & affirmativum

mativum se invicem alternatim sequuntur, *scil.* $+b+c-d+e\mp f+$ $+g-b+k$, &c. Atque sic Methodus nostra, non modo infinitarum Figurarum Quadraturas (ob indefinitam exponentem r ut in *Exemp.* 7, 8.) Sed etiam aliam Quadraturarum infinitatem uno simplicissimo & facillimo Calculo absolvit: Duæ enim hic existunt exponentes indefinitæ n, r, quarum quælibet infinities variari potest. Sed tamen multo generaliores reddi possunt, si non tantum Vinculi exponens (r), & quantitatis y extra vinculum (n); sed etiam exponens quantitatis y sub vinculo poneretur indefinita; quomodo autem hoc præstari possit, artificio postea tradendo explicabitur.

Exemp. 9. Inveniendæ sint Quadraturæ omnium Figurarum AMC, quarum naturæ exprimuntur hac æquatione $z = y + a \times \sqrt[r]{y+a}$: per Prob. 1. Æquatio definiens harum Quadratrices FGH (ubi PM=MC) est $dyy + eay + fa^2 \times \sqrt[r]{y+a} = xx$, ex qua pervenietur tandem (per Calculum sæpe explicatum) ad hanc æquationem.

$$\left.\begin{matrix} +dyy & +eay & +faa \\ +2rd & +re & +re \\ & +2rd & \end{matrix}\right\} = 2ryy + 4ray + 2raa.$$

Prima comparatio $d + 2rd = 2r$, unde $d = \dfrac{2r}{2r+1}$;

Secunda comparatio $e + re + 2rd = 4r$, unde $e = \dfrac{4rr + 4r}{\overline{2r+1} \times \overline{r+1}}$;

Tertia $f + re = 2r$, unde $f = \dfrac{2rr + 2r}{\overline{2r+1} \times \overline{r+1}}$;

Substitutis his coefficientium valoribus habetur Quadratura universalis quæsita; *scil.*

$$\text{Area} = \frac{ryy}{2r+1} + \frac{\overline{2rr+2r} \times ay + \overline{rr+r} \times aa}{\overline{2r+1} \times \overline{r+1}} \times \sqrt[r]{y+a} = \tfrac{1}{2}xx.$$

Exemp. 10.

Exemp. 10. Inveniendæ ſint Quadraturæ omnium Figurarum AMC quarum proprietates hæc æquatio $z = \overline{yy + ay} \times \sqrt[r]{y + a}$ includit: Erit $\overline{cy^3 + day^2 + eaay + fa^3} \times \sqrt[r]{y + a} = xx$; ex qua inventus valor lineæ PM (ſeu z) æquetur valori ejus dato, & pervenietur (ut ſupra) ad hac æquationem.

$$\left.\begin{array}{llll} +cy^3 & +dayy & +eaay & +faaa \\ +3rc & +2rd & +re & +re \\ & +3rc & +2rd & \end{array}\right\} = 2ryyy + 4rayy + 2raay.$$

Erit Prima comparatio $3rc + c = 2r$, unde $c = \frac{2r}{3r+1}$;

Secunda erit $d + 2rd + 3rc = 4r$, unde $d = \frac{6rr + 4r}{3r+1 \times 2r+1}$;

Tertia $e + re + 2rd = 2r$, unde $e = \frac{2rr + 2r}{3r+1 \times 2r+1 \times r+1}$;

Quarta denique comparatio $f + re = 0$, quæ poſt reductionem dabit $f = -\frac{2r^3 + 2r^2}{3r+1 \times 2r+1 \times r+1}$; ex quibus habetur

$$\text{Area} = \frac{ryyy}{3r+1} + \frac{\overline{3rr+2r} \times ay^2}{3r+1 \times 2r+1} + \frac{\overline{r^3+r} \times a^2y - \overline{r^3 - r^2} \times a^3}{3r+1 \times 2r+1 \times r+1} \times \sqrt[r]{y+a}$$

$= \frac{1}{2} xx$.

Ex duobus poſtremis Exemplis patet coefficientes, ut & æquationes quibus determinantur, ordine regulari procedere; eorum progreſſus in infinitum eodem modo, quo in ſuperiori Caſu invenitur. Sit ergo æquatio continens infinitam ſeriem Figurarum AMC, quarum Areæ ſunt determinandæ, talis $z = \overline{y^n + y^{n-1}} \times \sqrt[r]{y + a}$:

Erit

Erit per Problema noſtrum primum

$$\sqrt[r]{y+a} \times by^{n+1} + cy^{n} + dy^{n-1} + ey^{n-2} + fy^{n-3} \ \&c. = xx.$$

Ex qua inventus valor lineæ PM per Leibnitii Calculum adæquetur valori ejus dato ; & æquatio à fractis & ſurdis liberata dabit.

$$\begin{array}{llllll} \overline{n+1}\times rby^{n+1} & +\ n\times rcy^{n} & +\overline{n+}\times rdy^{n-1} & +\overline{n-2}\times rey^{n-2} & +\overline{n-3}\times & \\ \quad +b & +\overline{n+1}\times rb & +\ n\times re & +\overline{n-1}\times rd & +\overline{n-2}\times & \\ & \quad +c & +\ d & +\ e & + & \end{array}$$

$$\left.\begin{array}{l} \times rfy^{n-3} \\ \times re \quad \&c. \\ f \end{array}\right\} = 2\,ry^{n+1} + 4ry^{n} + 2\,ry^{n-1}.$$

Factaque debita comparatione terminorum hujus Æquationis erit.

$$b = +\frac{2r}{\overline{n+1}\times r+1}$$

$$c = +2r\times\frac{\overline{n+1}\times r+2}{\overline{n+1}\times r+1\times\overline{n\times r+1}}$$

$$d = +2r\times\frac{r+1}{\overline{n+1}\times r+1\times\overline{n\times r+1}\times\overline{n-1}\times r+1}$$

$$e = -2rr\times\frac{\overline{n-1}\times r+1}{\overline{n+1}\times r+1\times\overline{n\times r+1}\times\overline{n-1}\times r+1\times\overline{n-2}\times r+1}$$

$$f = +2r^{3}\times\frac{\overline{n-1}\times\overline{n-2}\times r+1}{\overline{n+1}\times r+1\times\overline{n\times r+1}\times\overline{n-1}\times r+1\times\overline{n-2}\times r+1\times\overline{n-3}\times r+1}$$

Ex compoſitione horum quinque terminorum apparet, quis ſit eorum progreſſus in infinitum, & ob continuam progreſſionem $\overline{n-1}\times\overline{n-2}\times\overline{n-3}$, *&c.* in numeratoribus patet quoque figuram eſſe quadrabilem quando (n) eſt numerus integer & poſitivus, totque terminos ſuis coefficientibus affectos aſſumendos eſſe, quot ſunt unitates in $n+2$: & quoad ſigna quibus connectuntur notandum tria prima eſt affirmativa, reliqua deinceps ſibi invicem alternatim ſuccedere.

cedere. Quæ omnia ex coefficientibus jam determinatis ſunt manifeſta. Idem quoque in aliis infinitis caſibus fieri poteſt, ſimilis enim eſt in omnibus proceſſus quem in his duobus explicaſſe ſuffecerit.

Cum plures ſuperſunt comparationes, quam quæ determinandis coefficientibus ſufficiunt, concludendum eſt Quadraturam quæſitam eſſe impoſſibilem, ſi valores coefficientium ex ſingulis comparationibus reperti non ſint ubique iidem; ſin ubique conveniant, tum Quadratura poſſibilis exiſtit. Ut ſi proponatur $z = y + a \times \sqrt[r]{yy + ay + a}$ æquatio definiens Curvas ACS; invenietur coefficientium (quæ curvarum FGH æquationem ingrediuntur) valores ex diverſis comparationibus eſſe diverſos, ac proinde Quadraturam Areæ AMC eſſe impoſſibilem: ſed ſi æquatio propoſita fuerit $z = 2y + a \times \sqrt[r]{yy + ay + a}$; invenietur eas ubique convenire, ac proinde Aream eſſe Quadrabilem. Et quidem univerſaliter, poſito $z = py + qa \times \sqrt[r]{yy + ay + a}$, ſi fuerit $p = 2q$, Area AMC ſemper quadrari poteſt, cujuſcunque tandem valoris ſupponatur vinculi exponens (r). Atque hujus modi Quadrabilitatis conditiones, ex Calculo noſtræ Methodi facili negotio inveniuntur, ut exempla ſequentibus conſtabit.

Exemp. II. Sit $z = py + qa \times \sqrt[r]{yy + ay + a}$, per Problema noſtrum generale erit $dyy + eay + fa^2 \times \sqrt[r]{y^2 + ay + a} = x^2$; ex qua inventus valor lineæ PM dabit tandem.

$$\left.\begin{array}{llll} 2\,dyyy & +2\,eay^2 & +2\,fa^2y & +fa^3 \\ 2\,rd & +2\,rd & +2\,rd & +re \\ & +re & +re & \\ & +d & +e & \end{array}\right\} = \begin{array}{llll} 2\,rpy^3 & +2\,rpay^2 & +2\,rpa^2y & +2rqa^3. \\ & +2\,rq & +2\,rq & \end{array}$$

Ex prima comparatione invenietur $d = \frac{rp}{r+1}$

Ex Secunda $e = \frac{2rrq + 2rq + rp}{r + 2 \times r + 1}$;

Ex Tertia $2f = \frac{rrp + 3rp + 2rq + 2rrq}{r + 2 \times r + 1}$;

Ex Quarta denique erit $2f = \frac{8rrq + 8rq - 2rrp}{r+2 \times r+1}$;

Manifeſtum itaque eſt, Figuram poſſe Quadrari, ſi valor quantitatis $2f$ fuerit idem ex tertia & quarta comparatione; fiat ergo æquatio inter utrumque, (omiſſo Denominatore utrinque communi) *ſcil.* $r^2p + 3rp + 2r^2q + 2rq = 8r^2q + 8rq - 2r^2p$. Quæ reducta dabit conditionem Quadrabilitatis, *ſcil.* $p = 2q$, quæ invenienda erat.

Adeo ut ſi $z = 2qy + qa \times \sqrt[r]{yy + ay + a}$; fuerit æquatio definiens curvas ACS, erit

$$\text{Area} = dyy + eay + faa \times \tfrac{1}{2}\sqrt[r]{yy + ay + a} = \tfrac{1}{2}xx.$$

In qua juxta determinationes jam inventas $d = \frac{2rq}{r+1}$, $2f = \frac{4r^2q + 8rq}{r+2 \times r+1}$, $e = \frac{2r^2q + 4rq}{r+2 \times r+1}$. Et ſpeciatim ſi ſit $q = 1$, $r = 2$, erit

$$\text{Area} = \tfrac{2}{3}yy + \tfrac{2}{3}ay + \tfrac{2}{3}aa \times \sqrt{yy + ay + a} = \tfrac{1}{2}xx.$$

Exemp. 12. Sit $z = pyy + qay \times \sqrt[r]{yy + ay + a}$. Erit rurſus per Problema primum generale

$$cy^3 + day^2 + ea^2y + fa^3 \times \sqrt[r]{y^2 + ay + a} = xx.$$

Ex qua inventus valor Lineæ PM dato ejus valori adæquatus dabit

$$\left.\begin{array}{lllll} +3rcy^4 & +3rcay^3 & +3rca^2y^2 & +2rda^3y & +rea^4 \\ +2c & +2rd & +2rd & +re & +f \\ & +2d & +re & +e & \\ & +c & +e & +2f & \\ & & +d & & \end{array}\right\} = \begin{array}{lll} 2rpy^4 & +2rpay^3 & +2rpay^2 \\ & +2rq & +2rq \end{array}$$

$+ 2rqay.$

Ex Prima comparatione $c = \frac{2rp}{3r+2}$;

Ex Secunda $d = \frac{3rrq + 2rq + rp}{3r+2 \times r+1}$;

Ex Tertia $e = \frac{2rrp + 3rp + 3rrq + 2rq}{3r + 2 \times r + 2 \times r + 1}$;

Ex Quarta comparatione $2f = \frac{3r^3q + 11r^2q + 6rq - 4r^3p - 9rrp - 3rp}{3r + 2 \times r + 2 \times r + 1}$;

Ex Quinta denique invenietur $2f = \frac{-4rrrp - 6rrp - 6rrrq - 4rrq}{3r + 2 \times r + 2 \times r + 1}$;

Unde patet Figurarum AMC (cujus Curva ACS propoſita æquatione definitur) non poſſe Quadrari, niſi numerator Fractionis ex Quarta, ſit æqualis numeratori Fractionis ex Quinta comparatione oriundæ. Facta itaque inter ipſos comparatione erit $3r^3q + 11r^2q + 6rq - 4r^3p - 9r^2p - 3rp = -4r^3p - 6r^2p - 6r^3q - 4rrq$; quæ reducta dabit $3rrq + 5rq + 2q = rp + p$. Nec unquam Quadrabitur Figura ſi capiatur p ad q in quavis alia ratione, quam ut $3rr + 5r + 2$ ad $r + 1$. Atque ſic inventa eſt Quadrabilitatis conditio, nec non ipſa Quadratura quæſita, *ſcil.*

$$\text{Area} = cy^3 + day^2 + eay + fa \times \tfrac{1}{2} \sqrt[r]{y^2 + ay + a} = \tfrac{1}{2} xx.$$

Exemp. 13. Sit $z = py^2 + qa \times \sqrt[r]{y^2 + ay + a}$ æquatio definiens Curvas ACS, ergo per Prob. 1.

$cy^3 + day^2 + ea^2y + fa^3 \times \sqrt[r]{yy + ay + a} = xx$. Ex qua per Methodum jam explicatum pervenietur ad hanc.

$$\left.\begin{array}{lllll} +3rcy^4 & +3rcay^3 & +3rcay^2 & +2rday & +rea \\ +2c & +2rd & +2rd & +re & +f \\ & +2d & +re & +2f & \\ & +c & +2e & +e & \\ & & +d & & \end{array}\right\} = \begin{array}{l} 2rpy^4 + 2rpay^3 + 2rpay^2 + \\ \qquad\qquad + 2rq \end{array}$$

$+ 2rqay + 2rqa.$

Ex prima comparatione invenietur $c = \frac{2rp}{3r + 2}$;

Ex Secunda erit $d = \frac{rp}{3r + 2 \times r + 1}$;

Ex Tertia $e = \frac{6r^3q + 10r^2q + 4rq + 2r^2p + 3rp}{3r + 2\times r + 2\times r + 1}$;

Ex Quarta erit $2f = \frac{6rrrq + 10r^2q + 3rq - 4r^3p - 9rrp - 3rp}{3r + 2\times r + 2\times r + 1}$;

Ex Quinta denique $2f = \frac{-6rrp - 4r^3p - 8r^2q - 20r^3q - 12r^4q}{3r + 2\times r + 2\times r + 1}$;

Facta itaque æquatione inter numeratores utriuſque valoris quantitatis $2f$, invenietur $p.\ q :: 12r^3 + 26rr + 18r + 4.\ 3r + 3$. Neque alia aſſignabilis eſt ratio p ad q, in qua propoſita æquatio definiat Figuram Quadrabilem. Atque hæc eſt conditio Quadrabilitatis inveniendæ; & quia inventæ ſunt coefficientes c, d, e, f; ideo inventa etiam eſt

$$\text{Area} = cy^3 + day^2 + eay + fa \times \tfrac{1}{2}\sqrt[r]{yy + ay + a} = xx.$$

Figuras *cum* Simpliciſſimis *ejuſdem Generis* *Comparare*.

QUando in æquatione aliqua Curvam definiente, quantitas abſciſſam denotans ad multas dimenſiones aſſurgit; tum Calculus in Methodo noſtra ſuperiori adhibendus non ſine aliqua laboris difficultate peragi poteſt: & quia computationis moleſtia, quantum Natura problematis patitur, ſummopere ſit evitanda; ideo ſubſidium hic adducam, quod hunc laborem vel prorſus auferret, vel ſaltem plurimum imminuet. Notum eſt Geometris, quomodo cuilibet Figuræ (ſive illa ſit Algebraica, ſive tranſcendens; ſive definitè, ſive indefinitè quadrabilis) aliæ infinitæ æquales inveniri poſſint. Si pariter conſtaret, quænam eſſent æquationes, quæ omnes Curvas definirent, quæ ejuſdem eſſent generis cum Curva Figuram quamlibet datam terminante; necnon quænam eſſet harum omnium ſimpliciſſima; tum ex hujus ſimpliciſſimæ Figuræ Quadraturâ, non modo datæ cujuſlibet, ſed omnium ſimilium Figurarum Quadraturæ innoteſcerent. Methodum, qua hoc præſtare ſoleo, breviter jam exponam, præmiſſo prius eleganti Theoremate ex Lectionibus Geometricis D. D. *Barrow* deſumpto, cujus demonſtrationem adjiciam, quam ipſe Author non addidit, eo quod ex ſuo demonſtrandi Modo facile deducitur.

LEMMA

LEMMA I.

Sint tres quælibet Curvæ BGL, OFN *(quarum axis* BD) DKM *(cujus axis* DL) *ita inter se relatæ, ut assumpto quovis puncto* G *in Curva* BG, *à quo ducantur, tangens* GA, *(cum axe concurrens in* A) *Ordinatæ* GC, CF *ad* LN; *item* GK *ad* BD *parallelæ, sit semper* AC. CG :: HK. CF. *Erit spatium* DHK *æquale spatio* BCFO, *nec non* DLM=BDNO, *& sic indefinite.*

Demonstratio.

Sit G*m* particula indefinite parva Curvæ BGL, ducantur *mr* ad GK, & (*ma*) ad GF parallelæ, secantes BD, GC, DL in punctis *e*, *n*, *c*. Jam quia Triangula G*nm*, GCA sunt similia, ideo AC. CG :: *mn*. G*n*, id est, AC. CG :: C*e*. H*c*. Sed ex hypothesi AC. CG :: HK. CF. ergo HK. CF :: C*e*. H*c*, unde HK×H*c*=CF×C*e*; cum vero idem de omnibus similibus rectangulis demonstrari possit, cumque spatium Curvilineum DHK, à summa omnium rectangulorum HK×H*c*; & spatium Curvilineum BCFO, à summa omnium rectangulorum CF×C*e* non differant; ergo DHK=BCFO. Q. D. E.

Schol. Si Linea DKM sit recta angulum faciens semirectum cum DL, tum coincidet hoc Theorema cum Lemmate primo, & proinde est hujus casus tantum particularis.

Notandum est, Quod datis Curvis BGL, DKM facile inveniatur reliqua OFN; vel datis BGL, OFN altera DKM similiter inveniri possit: tot itaque invenire possumus Figuras DHK æquales datæ cuilibet Figuræ BCFO, quot imaginari possumus Curvas BGL, id est, infinitas. Sed propositum nostrum est, solummodo Curvas DHK determinare, quæ sint similes, vel ejusdem generis cum data Curva OFN; ubi per Curvas similes, vel ejusdem generis intelligo eas, in quibus (ordinatim applicatis ad formam in probleme primo præscriptam reductis) Exponentes quantitatis abscissam denotantis habent easdem ubique relationes. His præmissis, sit

PRO-

PROB. II.

Fig. 2. *Data quacunque Curvâ Algebraicâ* DKM, *omnes Curvas* OFN *huic ſimiles invenire?*

IN ſuperiori Figura ſit communis abſciſſa BC$=x$. Ordinata CF$=$Y. Et Curvæ BGL ordinatæ GC$=z$, & quia GC$=$DH, erit etiam Curvæ alterius DKM; abſciſsâ DH$=z$, cujus ordinatim applicata HK$=u$. In æquatione Curvam datam DKM definiente, pro exponente vinculi particulari ponatur exponens indefinita (r); & afficiantur ejus termini coefficientibus l, p, q, s, &c. Erit perpetuo $z=x^m$ æquatio definiens Curvas ignotas BGL, cujus ope, & æquationis datæ invenietur (per *Lem.* 2.) æquatio definiens omnes Curvas OFN ſimiles datæ Curvæ DKM.

Exemp. 1. Sit æquatio $u=z^5+az\times\sqrt[2]{z^4+a}$ Datam curvam DKM determinans, & invenienda ſit alia, quæ omnes huic ſimiles determinet: Erit juxta præſcriptum Regulæ $u=lz^5+paz\times\sqrt[r]{qz^4+a}$, cum hac & aſſumpta $z=x^m$, quæſitam ſic invenies. Ex Analyticis valoribus linearum AC, CG, HK exterminentur indeterminatæ quantitates u, z per communes Algebræ regulas; tum cum his tribus & quarta CF$=y$ inſtituatur Analogia Lemmatis præcedentis, & hæc in æquationem mutata eſt quæſita. Sic per Tangentium Methodos invenietur AC$=\frac{x}{m}$, atqui patet GC$=x^m$, & ſubſtituendo x^m pro z in æquatione data (& coefficientibus ac vinculi exponente indefinitâ, affecta) erit HK$=lx^{5m}+pax^m\times\sqrt[r]{qx^{4m}+a}$.

Sed per *Lem.* 2. AC. CG :: HK. CF. hoc eſt in terminis Analyticis.

$$\frac{x}{m} : x^m :: lx^{5m}+pax^m\times\sqrt[r]{qx^{4m}+a} : y$$

Quæ Analogia, multiplicando terminos medios & extremos, dabit æquationem quæſitam, *ſcil.*

$$y=mlx^{6m-1}+mpax^{2m-1}\times\sqrt[r]{qx^{4m}+a}:$$

Quæ definit omnes Curvas OFN ſimiles curvæ DKM datæ. Simplicior

Simplicior tamen reddi poteſt, ſi pro exponente m ſubſtituatur alia exponens n diviſa per maximum diviſorem omnium numerorum, qui præfixi ſunt exponenti (m) in æquatione nuper inventâ. Sic quia 2 eſt maximus diviſor numerorum 6, 2, 4, ideo pono $m = \frac{1}{2}n$, unde $6m = 3n$, $2m = n$, $4m = 2n$, & proinde.

$$y = mlx^{3n-1} + mpax^{n-1} \times \sqrt[r]{qx^{2n} + a} :$$

Exemp. 2. Sit æquatio datam Curvam DKM definiens hujuſmodi $z^{39} \times \sqrt{z^{40} + a} = u$. Cum hac & aſſumpta $z = x^m$ invenietur per proceſſum præcedentis exempli $y = mpx^{40m-1} \times \sqrt[r]{x^{40m} + a}$: Et facta $m = \frac{1}{40}n$, erit $y = mpx^{n-1} \times \sqrt[r]{qx^n + a}$: Quæ definit omnes Curvas OFN ſimiles Curvæ datæ DKM, nam $n - 1 = 39$, & $n = 40$.

Prob. III.

Datâ quacunque Curvâ DKM, *omnium huic ſimilem ſimpliciſſimam determinare.*

In æquatione per *Problema* 2, inventa ponatur $n = 1$; & æquatio illa generalis in ſimpliciſſimam reſolvetur. Sic ſi in primo exemplo ponatur $n = 1$, erit $y = mlx^2 + pax \times \sqrt[r]{qx^2 + a}$. Quæ eſt ſimpliciſſima Curva OFN ſimilis datæ DKM. Pariter in ſecundo exemplo $y = mp\sqrt[r]{qx + a}$ eſt æquatio definiens ſimpliciſſimam Curvam OFN ſimilem Curvæ DKM per æquationem datam definitæ.

Prob. IV.

Ex datâ Quadraturâ Figuræ cujuſvis ſimpliciſſimæ BCFO, *alterius qualiſcunque ſimilis* DKH *Quadraturam determinare.*

In Quadratura Analytica Figuræ BCFO pro x ſubſtituatur z poteſtates habens ex *Problemate* 2 & 3, cognitas, eritque hæc Quadratura Figuræ DHK.

Exemp. 1. Invenienda ſit Quadratura Figuræ DHK, cujus

curvæ

curvæ proprietas ſit $u = z^7 \times \sqrt{z^8 + a}$: Ex hac & Aſſumpta $z = x^m$ invenietur per Problema 2, $y = mx^{8m-1} \times \sqrt{x^{8m} + a}$: & ponendo $8m = n$, Erit $y = mx^{n-1} \times \sqrt{x^n + a}$: Tum per Problema 3, fiat $n = 1$, & ſic $y = m\sqrt{x + a}$, eſt æquatio definiens ſimpliciſſimam Curvam OFN, cujus Area eſt æqualis Areæ Figuræ propoſitæ DHK. Sed per Methodum noſtram generalem $BCFO = \frac{x + a}{12} \times \sqrt{x + a}$. Jam quia $z = x^m$, & $8m = n = 1$, ideo $z = x^{\frac{1}{8}}$, ſeu $z^8 = x$. Subſtituta ita z^8 pro x in Quadratura Figuræ ſimpliciſſimæ jam inventa habetur $DHK = \frac{z^8 + a}{12} \times \sqrt{z^8 + a}$.

Exemp. 2. Sit $u = z^5 \times \sqrt{z^3 + a}$: proprietas Curvæ DKM, cujus Area invenienda ſit. Invenietur $y = mx^{6m-1} \times \sqrt{x^{3m} + a}$, & ponendo $3m = n$, erit $y = mx^{2n-1} \times \sqrt{x^n + a}$; & per Problema 3, fiat $n = 1$, unde $y = mx \times \sqrt{x + a}$; quæ definit ſimpliciſſimam Curvam OFN cujus Area (per *Lem.* 2.) eſt æqualis Areæ quæſitæ DHK. Jam verò per Methodum generalem invenio

$BCFO = \frac{2mxx}{5} + \frac{2mx - 4ma}{15} \times \sqrt{x + a}$: Quia $3m = n = 1$, ideo $m = \frac{1}{3}$ unde $z = x^{\frac{1}{3}}$, ſeu $z^3 = x$; ideoque $DHK = \frac{2z^6}{15} + \frac{2z^3 - 4a}{45} \times \sqrt{z^3 + a}$:

Notandum eſt hujuſmodi Arearum Quadraturas analyticè expreſſas non ſemper portioni abſciſſæ AM (*Fig.* 1.) vel BC aut DH (in hac *Fig.* 2.) competere, ſed eandem aliquando excedere, ut in penultimo; vel ab eadem deficere, ut in ultimo exemplo, quantitate quadam determinata, quæ Methodo inferius tradenda invenietur. Ob hanc rationem, in ſuperioribus exemplis poſui *Aream*, non vero AMC æqualem Quadraturis Analyticis ibi inventis: Quodque hic poſuerim BCFO, & DHK iiſdem æquales, diſtinctionis tantum gratiâ factum intellige, & in ſequentibus etiam hoc obſervandum erit.

Secundo notandum eſt, Quod in exemplis quarti Problematis, coefficientes l, p, q, s, &c. ut & exponentem indefinitam (r), juxta præſcriptum ſecundi Problematis adhibendas omiſerim, propterea quod in inveſtigatione alicujus particularis Figuræ DKH, ex ejuſdem generis ſimpliciſſima BCFO ſufficit æquationem Curvam

DKM

DKM definientem immutatam servare. Ex his constare arbitror, quam egregii sit usûs, hæc Figurarum cum simplicioribus comparatio, cum exinde ope unius Figuræ Quadratura ex Methodo nostra generali invenienda, habeatur alterius cujusvis, ejusdem generis & vinculi, Figuræ Quadratura. Longe tamen generalior futura est hæc Methodus, si, posita vinculi exponente (r) & adhibitis coefficientibus generalibus, l, p, q, s, &c. (ut in *Prob.* 2 & 3.) Ex una simplicissima Quadratura generali, omnium similium Figurarum Quadraturæ sub una generali Expressione determinentur.

Sit ergo in sequentibus DKH Figura simplicissima, cujus Curvæ DKM æquatio habeat Vinculi exponentem indefinitam r, infinitas proinde Curvas definiens, omnes tamen ejusdem simplicitatis, quoniam in nostra methodo, Calculus eadem facilitate procedit, sive magnus, sive parvus sit numerus r, sive etiam indefinitus statuatur. In Problematis enim Geometricis, quod factu facillimum est, illud simplicissimum haberi debet: in Quadraturis proinde, illæ Curvæ erunt simplicissimæ, quarum Areæ sunt Quadratu facillimæ; & ejusdem simplicitatis, quæ æque facilè Quadrari possunt. Sicut in constructione æquationum illæ Curvæ, quæ descriptu sunt facillimæ, etiam simplicissimæ habendæ sunt. Miror itaque Cartesium & alios, in eligendo Curvas Æquationum constructioni inservientes, non Descriptionis facilitatem, sed æquationum earum Naturas exprimentium compositionem respexisse. Sed hoc obiter. Sint etiam BCFO omnes Figuræ similes, Datæ Figuræ DKH.

Ex Methodi nostræ Exemplo 8, manifestum est, quod Area DKH sit quadrabilis quando in æquatione $u = sz^e \times \sqrt[r]{pz + a}$. Definiente Curvas DKM, exponens e est numerus integer & Positivus, aut nihilo æqualis. Sit ergo

§. 1. $e = 0$, unde $u = s\sqrt[r]{pz + a}$, ex hac & assumpta æquatione $z = x^m$ invenietur per Prob. 2 ; $y = msx^{m-1} \times \sqrt[r]{px^m + a}$; quæ definit omnes Curvas OFN, similes Curvæ DKM ; & DKH = BCFO per *Lem.* 2. atqui

$$DKH = \frac{rsz}{r+1} + \frac{rsa}{\overline{r+1} \times p} \times \sqrt[r]{pz + a}.$$

Subſtituendo itaque x^m pro z in hac data vel inventa Quadraturâ erit.

THEOR. I.

$$BCFO = \frac{rsx^m}{r+1} + \frac{rsa}{\overline{r+1} \times p} \times \sqrt[r]{px^m + a}.$$

Quod continet Quadraturas omnium Figurarum BCFO, quarum Curvæ OFN definiuntur æquatione generali per Prob. 2. inventâ.

§ 2. Sit $e=1$, unde $u = sz\sqrt[r]{pz+a}$: ex qua, cum aſſumpta $z=x^m$ per Prob. 2 invenietur, $y = msx^{2m-1} \times \sqrt[r]{px^m + a}$. Quæ definit omnes Curvas OFN ſimiles DKM, & in quibus DKH=BCFO. Sed

$$DKH = \frac{rszz}{2r+1} + \frac{rsaz}{\overline{2r+1} \times \overline{r+1} \times p} - \frac{rrsa^2}{\overline{2r+1} \times \overline{r+1} \times pp} \times \sqrt[r]{pz+a}.$$

Unde,

THEOR. II.

$$BCFO = \frac{rsx^{2m}}{2r+1} + \frac{arsx}{\overline{2r+1} \times \overline{r+1} \times p} - \frac{rrsa^2}{\overline{2r+1} \times \overline{r+1} \times pp} \times \sqrt[r]{px^m + a}.$$

§ 3. Sit $e=2$, unde $u = sz^2\sqrt[r]{pz+a}$: ex qua, cum æquatione aſſumpta $z=x^m$ invenietur per Prob. 2, $y = msx^{3m-1} \times \sqrt[r]{px^m + a}$, quæ definit omnes Curvas OFN ſimiles DKM, & in quibus DKH=BCFO, ſed

$$DKH = \frac{rsz^3}{3r+1} + \frac{rsaz^2}{\overline{3r+1} \times \overline{2r+1} \times p} - \frac{2r^2sa^2z}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1} \times pp}$$
$$+ \frac{2r^3sa^3}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1} \times ppp} \times \sqrt[r]{pz+a}:$$

THEOR.

THEOR. III.

$$BCFO = \frac{rsx^{3m}}{3r+1} + \frac{rsax^{2m}}{3r+1 \times 2r+1 \times p} - \frac{2r^2sa^2x^m}{3r+1 \times 2r+1 \times r+1 \times pp} +$$

$$+ \frac{2rrrsa^3}{3r+1 \times 2r+1 \times r+1 \times p^3} \times \sqrt[r]{px^m+a} :$$

§ 4. Sit $e=3$, unde $u=sz^3 \times \sqrt[r]{pz+a}$, quæ definit Curvas DKM, ex qua & æquatione aſſumpta $z=x^m$, per Problema ſecundum invenietur $y=msx^{4m-1} \times \sqrt[r]{px^m+a}$, quæ definit Naturas omnium Curvarum OFN, in quibus DKH=BCFO per Lem. 2. Sed

$$DKH = bz^4 + caz^3 + da^2z^2 + ea^3z + fa^4 \times \sqrt[r]{pz+a}, \text{ unde}$$

THEOR. IV.

$$BCFO = bx^{4m} + cax^{3m} + da^2x^{2m} + ea^3x^m + fa^4 \times \sqrt[r]{px^m+a}.$$

Quod continet Quadraturas omnium Figurarum BCFO, quarum Curvæ OFN præcedenti æquatione definiuntur. Atque hoc modo computari poſſunt Theoremata pro quibuſvis aliis exponentis e valoribus, nimirum $e=4$, $e=5$, $e=6$, &c. quouſque placet. Coefficientes b, c, d, e, f, in ultimo Theoremate per Methodum noſtram generalem determinantur, *ſcil.*

$$b = \frac{rs}{4r+1},\ c = \frac{rs}{4r+1 \times 3r+1 \times p},\ d = -\frac{3r^2s}{4r+1 \times 3r+1 \times 2r+1 \times pp},$$

$$e = \frac{6rrrs}{4r+1 \times 3r+1 \times 2r+1 \times r+1 \times ppp},\ f = -\frac{6rrrrs}{4r+1 \times 3r+1 \times 2r+1 \times r+1 \times ppp}$$

Ex his Theorematis habetur Quadratura omnium Figurarum, quarum Curvæ definiuntur per æquationes ad aliquam ſub hìc inventis contentam; quodnam vero ſit Theorema, cui particularis aliqua Figura referri debeat, ex relatione exponentium quantitatis x abſciſſam denotantis cognoſces, poſt reductionem ordinatæ ad debitam formam; nam quia $m-1$ in primo $2m-1$ in ſecundo, $3m-1$ in tertio, $4m-1$ in quarto, &c. ſunt exponentes quan-

titatis x extra vinculum; & ſola (m) in omnibus, exponens ejuſdem ſub vinculo: Exinde patet, quod ſi exponenti extra vinculum addatur unitas, & ſumma per exponentem (quantitatis x) ſub vinculo dividatur; quando Quotiens eſt 1, aſſumendum erit primum; quando Quotiens eſt 2, aſſumendum erit ſecundum; quando 3, tertium; & ſic porro in infinitum. Cognito (hoc modo) Theoremate, Figuræ alicujus Quadraturam includente; fiat debita comparatio ordinatæ particularis, cum ordinatæ generalis valore, & ſic innoteſcent m, r, s, p, q. Quorum valores particulares ſic inventi in Theoremate ſubſtituti dabunt Quadraturam quæſitam.

Exemp. 1. Invenienda ſit Quadratura parabolæ cujus notiſſima æquatio eſt $y = \sqrt[2]{cx}$. Quia $\frac{0+1}{1} = 1$, ideo Quadratura Parabolæ in Theoremate primo continetur. Fiat ergo comparatio inter ordinatæ generalis, (ad primum Theorema ſpectantis) & particularis in Parabola Ordinatæ valores; ex qua comparatione invenietur $r=2$, $m=1$, $s=1$, $p=c$, $a=0$ hi valores quantitatum r, m, s, p, a, in Theoremate primo ſubſtituti dabunt Parabolæ Aream $=$

$$= \frac{2x}{3} \times \sqrt{cx}.$$

Exemp. 2. Invenienda ſit Quadratura Figuræ, cujus proprietas ſit $y^2c^2 - 2y^2xx = 16x^6$, Ordinata ad præſcriptam formam reducta erit $y = 4x^3 \times \sqrt[-2]{cc - 2x^2}$. Quia $\frac{3+1}{2} = 2$, ideo Quadratura hujus Figuræ in Theoremate ſecundo includitur. Factâ itaque comparatione inter hujus Figuræ datæ, & generalis (ad Theor. 2. ſpectantis) Ordinatæ valorem, invenietur $r=-2$; $m=2$; $ms=4$; ſeu $s=2$; $p=-2$; $a=cc$. Quibus in Theoremate 2 ſubſtitutis, erit Area Figuræ propoſitæ

$$= \frac{12x^4 - 2ccx^2 - 2c^4}{15} \times \sqrt[-2]{c^2 - 2x^2} = \tfrac{2}{15}\sqrt{c^6 + 4c^4x^2 - 3c^2x^4 - 18x^6}.$$

Exemp. 3. Invenienda ſit Quadratura Figuræ, cujus proprietas ſit $y = x^5 \times \sqrt{x^2 - c}$: Quia $\frac{5+1}{2} = 3$, ideo hæc Figura ad Theor. 3. eſt referenda; facta itaque comparatione Ordinatæ particularis datæ, & generalis eam includentis, invenietur $r=2$, $m=2$, $s=1$, $p=1$, $-a=c$; quibus in Theor. 3 ſubſtitutis, erit quæſita

Area

Area $= \frac{2}{7} x^6 - \frac{2}{35} cx^4 - \frac{8}{105} c^3 x^2 - \frac{16}{105} c^3 \times \sqrt{x^2 - c}$:

Exemp. 4. Invenienda ſit Quadratura Figuræ, cujus Curva ſit $yc^4 - 2yc^2 x^2 + yx^4 = 4x$. Ordinata ad præſcriptam formam reducta, Erit $y = 4x \sqrt{-x^2 + c^2}^{\,-\frac{1}{2}}$: Quia $\frac{1+1}{2} = 1$, Ideo hæc Figura ad Theor. 1, referenda eſt, & facta debita comparatione ut in præcedentibus invenietur $r = -\frac{1}{2}$, $s = 2$, $m = 2$, $p = -1$, $a = c^2$, Quibus in Theor. 1, ſubſtitutis, erit quæſita

$$\text{Area} = -2x^2 + 2c^2 \times \sqrt{-x^2 + c^2}^{\,-\frac{1}{2}} = \frac{-2x^2 + 2cc}{c^4 - 2c^2x^2 + x^4} = \frac{2}{c^2 - x^2}.$$

In Exemplo decimo oſtenſum eſt, Aream ſemper poſſe Quadrari, quando in æquatione $u = sz^n + paz^{n-1} \times \sqrt[r]{qz + a}$; exponens n eſt Numerus integer & affirmativus: unde alia prodibit ſeries Theorematum, ex vario quantitatis n valore dependentium. Sit itaque

§ 5. $n = 1$, unde $u = sz + pa \times \sqrt[r]{qz + a}$ eſt æquatio definiens Curvas DKM, ex qua & aſſumpta $z = x^m$ invenietur per Problema ſecundum $y = msx^{2m-1} + mpax^{m-1} \times \sqrt[r]{qx^m + a}$, quæ exprimit Naturas omnium Curvarum OFN, ſimiles Curvis DKM; & in quibus per Lem. 2, DKH = BCFO, ſed DKH $= bzz + caz + daa \times \sqrt[r]{qz + a}$. Subſtituendo x^m pro z, erit

Theor. V.

$$\text{BCFO} = bx^{2m} + cax^m + daa \times \sqrt[r]{qx^m + a}:$$

Quod continet Quadraturas omnium Figurarum BCFO ſimilium Figuris DHK: Quantitates b, c, d, per Methodum noſtram generalem ſic determinatas inveni.

$$b = \frac{rs}{2r+1}; \quad c = \frac{2rrpq + rpq + rs}{2r+1 \times r+1 \times q}; \quad d = r \times \frac{2rpq + pq - rs}{2r+1 \times r+1 \times qq};$$

§ 6. Sit

§ 6. Sit $n=2$, unde $u=szz+paz\times\sqrt[r]{qz+a}$ est æquatio definiens Curvas DKM, ex qua & assumpta $z=x^m$ (definiente semper Curvas BGL) invenietur per Prob. 2.

$$y=msx^{3m-1}+mpax^{2m-1}\times\sqrt[r]{qx^m+a}.$$

Quæ definit Naturas omnium Curvarum OFN, ejusdem generis cum Curvis DKM, in quibus DKH=BCFO per Lem. 2. sed DHK$=bz^3+caz^2+daaz+ea^3\times\sqrt[r]{qz+a}$; posito x^m pro z, erit

THEOR. VI.

BCFO$=bx^{3m}+cax^{2m}+daax^m+ea^3\times\sqrt[r]{qx^m+a}$; & in utraque $b=\frac{rs}{3r+1}$; $c=\frac{3rrpq+rpq+rs}{3r+1\times 2r+1\times q}$; $d=rx\frac{3rpq+pq-2rs}{3r+1\times 2r+1\times r+1\times qq}$; $e=-rr\times\frac{3rpq+pq-2rs}{3r+1\times 2r+1\times r+1\times qqq}$.

§ 7. Sit $n=3$, unde $u=sz^3+paz^2\times\sqrt[r]{qz+a}$ est æquatio definiens DKM, ex qua & assumpta $z=x^m$ invenietur per Problema secundum, $y=msx^{4m-1}+mpax^{3m-1}\times\sqrt[r]{qx^m+a}$ æquatio definiens omnes similes Curvas OFN, in quibus DHK=BCFO per Lem. 2. Sed DHK$=bz^4+caz^3+da^2z^2+ea^3z+fa^4\times\sqrt[r]{qz+a}$: ponatur x^m pro z.

THEOR. VII.

BCFO$=bx^{4m}+cax^{3m}+da^2x^{2m}+ea^3x^m+fa^4\times\sqrt[r]{qx^m+a}$.

in quo $b=\frac{rs}{4r+1}$; $c=\frac{4rrpq+rpq+rs}{4r+1\times 3r+1\times q}$; $d=rx\frac{4rpq+pq-3rs}{4r+1\times 3r+1\times 2r+1\times q^2}$; $e=-r^2\times\frac{8rpq+2pq-6rs}{4r+1\times 3r+1\times 2r+1\times r+1\times q^3}$; $f=+r^3\times\frac{8rpq+2pq-6rs}{4r+1\times 3r+1\times 2r+1\times r+1\times q^4}$

§ 8. Sit

§ 8: Sit $n=4$, unde $u=sz^4+paz^3\times\sqrt[r]{qz+a}$ eſt æquatio definiens Curvas DKM; ex qua cum aſſumpta $z=x^m$ invenietur per Prob. 2. $y=msx^{5m-1}+mpax^{4m-1}\times\sqrt[r]{qx^m+a}$ æquatio definiens omnes ſimiles Curvas OFN, in quibus DHK=BCFO per Lem. 2. Sed DHK$=bz^5+caz^4+da^2z^3+ea^3z^2+fa^4z+ga^5\times\sqrt[r]{qz+a}$: Ergo;

THEOR. VIII.

BCFO$=bx^{5m}+cax^{4m}+da^2x^{3m}+ea^3x^{2m}+fa^4x^m+ga^5\times\sqrt[r]{qx^m+a}$:

In quo $b=\frac{rs}{5r+1}$; $c=\frac{5r^2pq+rpq+rs}{5r+1\times 4r+1\times q}$; $d=r\times\frac{5rpq+pq-4rs}{5r+1\times 4r+1\times 3r+1\times q^2}$;

$e=-r^2\times\frac{15rpq+3pq-12rs}{5r+1\times 4r+1\times 3r+1\times 2r+1\times q^3}$;

$f=r^3\times\frac{30rpq+6pq-24rs}{5r+1\times 4r+1\times 3r+1\times 2r+1\times r+1\times q^4}$;

$g=-r^4\times\frac{30rpq+6pq-24rs}{5r+1\times 4r+1\times 3r+1\times 2r+1\times r+1\times q^5}$;

Ex his patet, non modo æquationum Curvas OFN, ſed etiam Theorematum, earum Areas BCFO determinantium progreſſus in infinitum; & facile eſt innumeras alias Theorematum ſeries formare. Pro hujuſmodi Figuris, quarum Valor Analyticus ordinatim applicatarum ita reduci poſſunt, ut duobus tantum exiſtentibus terminis ſub Vinculo, duo etiam in illos multiplicati, extra vinculum exiſtant. Ex Methodo mea generali Exemplis 8 & 10 illuſtrata, inveni Aream DHK poſſe Quadrari quando in æquatione $u=sz^n+pa^cz^{n-c}\times\sqrt[r]{qz+a}$ Naturam Curvæ DKM definiente, exponentes n, c ſunt numeri integri & poſitivi, poſito $n \sqsubset c$. Præcedens Theorematum ſeries, quatuor ultimis Sectionibus inventa, eſt Caſus omnium primus, in quo nempe $c=1$; Caſum ſecundum (in quo $c=2$) adjiciam; cuique viſum fuerit reliquos eodem modo proſequatur.

Exiſtente itaque $c=2$, erit $u=sz^n+paaz^{n-2}\times\sqrt[r]{qz+a}$: unde per Methodum noſtram generalem invenietur

Area

Area $= bz^{n+1} + caz^{n} + da^{2}z^{n-1} + ea^{3}z^{n-2} + fa^{4}z^{n-3} + ga^{5}z^{n-4}$, &c.

$\times \frac{1}{2} \sqrt[r]{qz+a}$:

In qua $b = \frac{2rs}{n+1 \times r+1}$; $c = \frac{2rs}{n+1 \times r+1 \times n \times r+1 \times q}$;

$d = \frac{n+1 \times r+1 \times n \times r+1 \times 2rpq^{2} - 2nrrs}{n+1 \times r+1 \times n \times r+1 \times n-1 \times r+1 \times qq}$; $e = \frac{2rp - n-1 \times rd}{n-2 \times r+1 \times q}$;

$f = -\frac{n-2 \times re}{n-3 \times r+1 \times q}$; $g = -\frac{n-3 \times rf}{n-4 \times r+1 \times q}$;

Eodem modo ſequens terminus erit $ha^{6}z^{n-5}$, in quo

$h = -\frac{n-4 \times rg}{n-5 \times r+1 \times q}$.

Ex his patet, horum progreſſus in infinitum: & ob continuam multiplicationem ex $n-2 \times n-3 \times n-4$, &c. in numeratoribus coefficientium; ideo ſi ſit $n=2$, tum e erit ultimus; ſi $n=3$, f erit ultimus; ſi $n=4$, ergo g erit ultimus Terminus, qui in $\frac{1}{2}\sqrt[r]{qz+a}$ multiplicatus conſtituet Quadraturam quæſitam Areæ DKH.

Determinatis generaliter coefficientibus, Theorematum computatio, nullo fere negotio computatur.

§ 9. Sit $n=2$, unde $u = sz^{2} + paa \times \sqrt[r]{qz+a}$: ex qua cum æquatione aſſumpta $z = x^{m}$, invenietur $y = msx^{3m-1} + mpaa \times \times \sqrt[r]{qx^{m}+a}$: quæ exprimit nuturas omnium Curvarum OFN ſimilium Curvis DKM, & in quibus (per Lem. 2.) DHK = BCFO.

Sed DHK $= bz^{3} + caz^{2} + da^{2}z + ea^{3} \times \sqrt[r]{qz+a}$;

THEOR.

THEOR. IX.

$$BCFO = bx^{3m} + cax^{2m} + da^2x^m + ea^3 \times \sqrt[r]{qx^m + a}:$$

In quo (juxta præcedentes coefficientium determinationes generales)

$$b = \frac{rs}{3r+1};\ c = \frac{rs}{3r+1 \times 2r+1 \times q};\ d = \frac{6r^3pq^2 + 5r^2pq^2 + rpq^2 - 2r^2s}{3r+1 \times 2r+1 \times r+1 \times q^2};$$

$$e = \frac{6r^3pq^2 + 5r^2pq^2 + 2r^3s}{3r+1 \times 2r+1 \times +1 \times q^3}.$$

§ 10. Sit $n=3$, unde $u = sz^3 + pa^2z \times \sqrt[r]{qz + a}$: Ex qua & assumpta $z = x^m$ invenietur per Prob. 2. $y = msx^{4m-1} + mpaax^{2m-1} \times \sqrt[r]{qx^m + a}$: quæ definit omnes Curvas OFN, quarum exponentes abscissarum z, x easdem habent relationes. Sed

$$DHK = bz^4 + caz^3 + da^2z^2 + ea^3z + fa^4 \times \sqrt[r]{qz + a}:$$

THEOR. X.

$$BCFO = bx^{4m} + cax^{3m} + da^2x^{2m} + ea^3x^m + fa^4 \times \sqrt[r]{qx^m + a}.$$

In quo (juxta generales coefficientium determinationes § 9. præmissas)

$$b = \frac{rs}{4r+1};\ c = \frac{rs}{4r+1 \times 3r+1 \times q};\ d = \frac{12r^3pq^2 + 7r^2pq^2 + rpq^2 - 3r^2s}{4r+1 \times 3r+1 \times 2r+1 \times qq};$$

$$f = -\frac{12r^4pq^2 + 7r^3pq^2 + r^2pq^2 + 6r^4s}{4r+1 \times 3r+1 \times 2r+1 \times r+1 \times q^4};$$

$$e = \frac{12r^3pq^2 + 7r^2pq^2 + rpq^2 + 6r^3s}{4r+1 \times 3r+1 \times 2r+1 \times r+1 \times q^3};$$

Jamque ea omnia complexus sum, quæ spectant ad Quadraturas Figurarum BCFO, deducibiles ex similibus Figuris

DHK, quarum ordinatæ Analyticè expreſſæ ſint $u = sz^{n} + pa^{e}z^{n-e} \times \sqrt[r]{qz + a}$: exiſtente z unius dimenſionis ſub vinculo. Et pro aliis ſuppoſitis intra vinculum ejus dimenſionibus, alia infinita Theoremata computari poſſunt: plerumque tamen non ſine aliquibus Quadrabilitatis reſtrictionibus: nullas certè hactenus tractare contigit, præter jam memoratas, quæ abſolutè Quadrari poſſunt, quando Ordinata duos extra vinculum terminos, in totidem ſub vinculo multiplicatos habet. Theorema unum aut alterum, Ex Figuris DHK habentibus z^2 ſub vinculo deducta adjiciam.

§ 11. Sit $u = sz^2 + pa^2 \times \sqrt[r]{qz^2 + a^2}$ æquatio definiens Curvas DKM, ex qua cum Aſſumpta $z = x^m$, invenietur per Problema 2.

$y = msx^{3m-1} + mpa^2x^{m-1} \times \sqrt[r]{qx^{2m} + a^2}$: quæ definit omnes Curvas OFN ſimiles Curvis DKM. Sed

$$DHK = \frac{rsz^3}{3r+2} + pa^2z^2 \times \sqrt[r]{qz^2 + a^2}.$$ Unde

THEOR. XI.

$$BCFO = \frac{rsx^{3m}}{3r+1} + pa^2x^m \times \sqrt[r]{qx^{2m} + a^2}.$$

Et per Methodum noſtram exemplis 11, 12, 13, explicatam invenietur Quadrabilitatis conditio $s = \frac{3rpq + 2pq}{r}$.

§ 12. Sit $u = sz^4 + pa^2z^2 \times \sqrt[r]{qz^2 + a^2}$. Ex qua tum aſſumpta æquatione invenietur per Problema 2.

$$y = msx^{5m-1} + mpa^2x^{3m-1} \times \sqrt[r]{qx^{2m} + a^2}.$$

Quæ definit omnes Curvas OFN ſimiles, Curvis DKM. Sed

$DHK = \frac{rsz^5}{5r+2} + \frac{pa^2z^3}{3} \times \sqrt[r]{qz + a^2}$: Et ſubſtituo x^m pro z, erit.

THEOR. XII.

$$BCFO = \frac{rsx^{5m}}{5r+2} + \frac{pa^3x^{3m}}{3} \times \sqrt[r]{qx^{2m}+a^2} :$$

Quadrabilitatis conditio Methodo ſuperiori inventa, eſt $s = \frac{5rpq+2pq}{3r}$. Obiter hic notari velim Figuram illam, cujus Quadraturam exhibuit Dominus $D. T.$ in Actis eruditorum, ad hoc Theorema reduci. Æquatio definiens ejus Curvam eſt $y^2 = \frac{9ccx - 12cx^2 + 4x^3}{2c - x}$. Reducta itaque Ordinata ad formulam Sectionis 12, erit $y = -2x^{\frac{3}{2}} + 3cx^{\frac{1}{2}} \times \sqrt[-2]{-x+2c}$: ex comparatione debita hujus, cum generali illam includente, patet $2m = 1$, $3m - 1 = \frac{1}{2}$, $5m - 1 = \frac{3}{2}$, ex quibus $m = \frac{1}{2}$; ſimiliter $r = -2$, $aa = 2c$, $q = -1$; $ms = -3$, unde $s = -4$; $mpa^2 = 3c$, unde $p = 3$. His valoribus quantitatum m, r, a^2, q, s, p. In Theoremate generali ſubſtitutis, invenietur

$$Area = -x^{\frac{5}{2}} + 2cx^{\frac{3}{2}} \times \sqrt[-2]{-x+2c} = \frac{-\sqrt{x^5} + 2c\sqrt{x^3}}{\sqrt{2c-x}} = \sqrt{2cx^3 - x^4} :$$

Tertius Figurarum Ordo hic eſt, in quo ordinatim applicata tres habet terminos extra vinculum multiplicatos in duos ſub vinculo; pro quibus Theoremata generalia computantur ut in præcedentibus. Exempli gratia,

§ 13. Sit $u = sz^2 + paz + la^2 \times \sqrt[r]{qz+a}$, æquatio definiens DKM, ex qua cum aſſumpta $z = x^m$ invenietur per Problema 2. $y = msx^{3m-1} + mpax^{2m-1} + mla^3x^{m-1} \times \sqrt[r]{qxm+a}$; quæ definit omnes Curvas OFN ejuſdem generis cum DKM, & in quibus DKH = BCFO per Lem. 2. Sed

$$DKH = bz^3 + caz^2 + da^2z + ea^3 \times \sqrt[r]{qz+a} ;\ \text{unde}$$

Theor. XIII.

$$BCFO = \overline{bx^{3m} + cax^{2m} + da^2 x^m + ea^3} \times \sqrt[r]{\overline{qx^m + a}}.$$

In quo $b = \frac{rs}{3r+1}$; $c = \frac{6r^2pq + 2rp + 2rs}{\overline{3r+1} \times \overline{r+1} \times 2q}$;

$d = \frac{\overline{3r+1} \times \overline{2r+1} \times rlq^2 + 3r^2pq + rpq - 2r^2s}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1} \times q^2}$;

$e = \frac{\overline{6r^3 + 5r^2 + r} \times lq^2 - 3r^3pq - r^2pq + 2r^3s}{\overline{3r+1} \times \overline{2r+1} \times \overline{r+1} \times q^3}$;

Facile eſſet horum Theorematum ſeries, quouſque placet, continuare, nec non eorum progreſſum in infinitum detegere: Ex Methodo enim generali ſtatim apparet, hunc Figurarum ordinem ſemper poſſe quadrari, quando $u = \overline{sz^n + paz^{n-1} + la^2 z^{n-2}} \times \sqrt[r]{\overline{qz+a}}$, modo n ſit numerus integer & affirmativus. Atque ſic progrediendum eſt ad altiores Figurarum ordines, in quibus ordinata habet quatuor, quinque, ſex, *&c.* terminos extra vinculum, multiplicatos in duos terminos ſub vinculo incluſos.

Atque jam Methodum hanc pluſquam ſatis explicuiſſe ſpero, ita ut illius proceſſus in Figuris, quarum ordinatæ habent tres, quatuor, quinque, *&c.* terminos ſub vinculo, in quoſdam extra illud multiplicatos, ulteriori explicatione non indigeat. Hic candidè fateri æquum eſt, non ſemper hoc modo Figuras cum abſolutè ſimpliciſſimis comparari. Atqui me, quod ſuſceperam, præſtitiſſe ſufficiat, Methodum exhibendo, qua Figuræ cum ſimpliciſſimis ejuſdem generis comparari poſſint.

Hic vero ſpeculatio notatu digniſſima occurrit, nimirum non ſemper dari immediatum regreſſum à Figura propoſita, ad ſimpliciſſimam, quacum illius Area comparari poteſt; ſed interdum eſſe duas, interdum tres, interdum quatuor, *&c.* intermedias Figuras, quarum prima ſimplicior eſt quam propoſita, & ſecunda quam prima, tertia quam ſecunda, quarta quam tertia, & ſic deinceps, donec tandem ad omnium abſolutè ſimpliciſſimam peveniatur. Harum Figurarum, à ſimplicioribus ad magis magiſque in infinitum compoſitas, progreſſus ex Lem. 2. facillimè ſic deducitur.

In

In adjecta Figura sint tres quælibet Curvæ *bgl*, *bfn* (quarum axis *bd*, Ordinatæ *ld*, *dn*) *dkm* (cujus axis *dl*) ita inter se relatæ, ut ductis à quovis puncto *g* (in Curva *bgl*) tangente *ga*, necnon *gcf*, *ghk* parallelis ad *ldn* & *bd*; sit perpetuo. *ac*. *cg* : : *hk*. *cf*. Tum à puncto *b* ducatur *box* parallela ad *ldn*, sintque duæ aliæ Curvæ *ozd*, *oi*S (quarum axis *ob*) ita ad præcedentem Curvam *bfn* relatæ, ut producta *fcg*, donec occurrat Curvæ *ozd* in puncto *z*, à quo ducatur Tangens *zx*, & *zpi* ad *bcd* parallela, sit perpetuo *xp*. *pz* : : *cf*. *pi*. Tum à puncto *o* ducatur *otq* ad *bcd* parallela, sintque duæ aliæ Curvæ *trb*, *tu*T ita ad præcedentem *oi*S relatæ, ut producta *ipz* donec occurrat Curvæ *trb*, in puncto *r*, & ductis, Tangente *rq*, ordinatis *rs*, *su*, sit semper *qs*. *sr* : : *pi*. *su*. His, inquam, positis, erit *tsu* = *opi* = *bcf* = *dhk*; quarum *opi* est simplicior quam *tus*, & *bcf* simplicior quam *opi*, & *dhk* simplicior quam *bcf*. Atque sic alias Figuras, magis quam *tus* gradatim compositas invenies: earumque Naturas æquationibus Algebraicis definire licet, data nimirum prima Figura DHK, necnon datis vel assumptis æquationibus Curvarum *bgl*, *ozd*, *trb*, ad quas nempe ducuntur tangentes *ag*, *zx*, *qr*. Exempli gratia. Sic notatis quantitatibus, $bc = pz = x$, $cg = dh = z$, $cf = y$, $kh = u$, $op = sr = v$, $pi = p$, $ts = w$, $su = s$. Sit $u = \sqrt{az + a}$, $z = x^2$, & $x = v^2 + av$, & $v = w^2$; ex his per Analogias ante positas invenietur, $y = 2x\sqrt{ax^2 + a}$, quæ definit Curvam *bfn*, & $p = 4v^3 + 6av^2 + 2a^2v \times \sqrt{av^4 + 2a^2v^3 + a^3v^2 + a}$. Quæ definit Curvam *oi*S; & deniq; $s = 8w^7 + 12aw^5 + 4a^2w^3 \times \sqrt{aw^8 + 2a^2w^6 + a^3w^4 + a}$; quæ definit Curvam *tu*T. Et quamvis ejus æquatio sit per quam composita, tamen patet illius Quadraturam, ex Parabolæ *dhk* Quadratura dependere; ita ut hac cognita, illa pariter innotescat, modo daretur regressus à Curva *u*T, ad Curvam *oi*S, & ab *oi*S ad Curvam *bfn*, & tandem à *bfn* ad parabolam *dkm*. Esset quidem, hoc, aliquid in Geometria, Algebræ Analogum præstare; sicut in hac, ex quantitatibus quibusdam datis, per varios æquationum resolutionis gradus, ex una in aliam fit transitus, donec tandem in æquationem omnino cognitam perveniatur; sic in illa, ex data Curva *tu*T, (cujus Area *tus* est incognita) per varias intermedias Curvas *oi*S, *bfn* fieret transitus, donec tandem ad Curvam *dkm* cognitæ Quadraturæ perveniretur. Hanc itaque nobilissimam speculationem, iis, qui eam pro sua dignitate tractare possunt, relinquo.

Atque jam Methodi hujus partem priorem me absolvisse puto, si pauca addidissem, ad Quadraturam Expressiones Analyticas spectantia.

De

De Quadraturarum *Expressione Analytica.*

JAM præmonui Arearum Expressiones Analyticas nostra Methodo inventas, ab initio Abscissæ non semper computari, sed ab ea aliquando deficere, & eandem aliquando excedere quantitate quadam determinata. Notandum itaque punctum illud, à quo Areæ computantur, esse interdum supra initium Abscissæ, interdum in ipso ejus initio, interdum etiam infra illud; & non rarò prorsus imaginarium esse. Distinctionis gratia, cui expressio sic inventa competit, licebit *punctum simplicissimæ Expressionis* appellare.

Notandum secundo, quod si Areæ Expressio Analytica in se contineat terminum determinatum, tum infallibiliter ab initio Abscissæ non computetur: Sin indeterminata quantitas Abscissam designans omnes ejus terminos afficiat, tum præcisè Areæ Abscissæ adjacenti conveniat. Duo itaque hic præstanda sunt; primo, Data quavis Areæ Expressione Analytica, punctum, à quo computatur, invenire. Secundo, Dato puncto simplicissimæ Expressionis, invenire Expressionem Abscissæ convenientem.

Cum totius Methodi nostræ fundamentum in eo positum sit, ut inveniatur Curva FGH, cujus intercepta PM sit æqualis ordinatim applicatæ MC in Figura Quadranda AMC. Proinde si Geometricè describatur Curva FGH, (quam ob usum suum *Quadraticem* in posterum vocabo) ex illius relatione ad Quadrandam AMC, hæc duo quæsita statim innotescunt.

Assumatur itaque casus particularis Exempli 9, in quo $r = 2$, unde $z = y + a \times \sqrt{y+a}$; est æquatio definiens Curvam NC, in qua abscissa AM $= y$, ordinata MC $= z$, & AN $= a\sqrt{a}$. Atque $\frac{4y^2 + 8ay + 4a^2}{5} \times \sqrt{y+a} = xx$ æquatio definiens Quadratricem GFH, quæ Geometricè descripta cum axe concurrit in puncto H, supra initium abscissæ A. Dico punctum illud H, esse punctum simplicissimæ expressionis, & proinde $\frac{1}{2}$ GM $q = \frac{1}{2} xx$ non esse expressionem Areæ AMCN abscissæ AM competentis, sed, continuata

nuata Curva NC ad H, esse Aream HMC; adeoque excedere Aream abscissæ adjacentem, toto spatio trilineo HAN $= \frac{1}{2}$ FAq.

Secundo, Assumatur Exemplum secundum, in quo $z = y\sqrt{y+a}$: definit Quadrandam AMC, & $\frac{12y^2+4ay-8a^2}{15} \times \sqrt{y+a} = xx$ Quadraticem FHG, quæ Geometricè descripta concurret cum axe in puncto simplicissimæ expressionis H infra initium abscissæ A, ita ut integra Quadratrix sit FHG, in qua, crescentibus abscissis, decrescunt ordinatæ, donec in puncto concursûs H prorsus evanescant. Dein ab hoc puncto H, crescentibus abscissis, crescunt pariter ordinatæ usque in infinitum. Patet itaque $\frac{1}{2}$ GM$q = \frac{1}{2}$ xx non integræ Areæ AMC, sed illius tantum parti HMCN competere; adeoque Expressio superius inventa deficit ab ea, quæ Abscissæ AM adjacet, toto spatio trilineo HAN $= \frac{1}{2}$ AFq. Fig. 5.

Assumatur tertio, Exemplum primum in quo $z = y\sqrt{y^2+a^2}$ Quadrandam AMC, & $\frac{2y^2+2a^2}{3} \times \sqrt{y^2+a^2} = x^2$ Quadratricem FG definiunt; hæc autem Geometricè descripta nusquam cum axe concurrit, sed ab eodem (continuata) versus K deflectit: Quo casu punctum simplicissimæ expressionis merè imaginarium est. Patet itaque $\frac{1}{2}$ GM$q = \frac{1}{2}x^2$, non competere Areæ AMC abscissæ AM adjacenti, sed eandem excedere toto spatio $\frac{1}{2}$ FAq. Hæc omnia ex Lemmate primo tam evidenter sequuntur, ut iis demonstrandis inhærere superfluum esset. Quæque de his tribus Figuris dicta sunt; omnibus aliis facillimè applicari possunt. Superest tantum, ut ostendam quo pacto, punctum simplicissimæ Expressionis H, necnon $\frac{1}{2}$ FAq spatium deficiens vel excedens Aream quæsitam AMC Analyticè inveniatur. Fig. 6.

Ex dictis manifestum est punctum simplicissimæ expressionis H, illud esse in quo Quadratrix cum axe concurrit, id est, ubi ordinatim applicatæ x evanescunt, seu nihilo fiunt æquales: Et proinde si in æquatione Quadratricem definiente ponatur $x = o$, hæc resoluta dabit longitudinem abscissæ y, juxta quam Quadratrix cum axe concurrit, quod punctum est simplicissimæ expressionis H quæsitum: quod si valor quantitatis y sic inventus, fuerit affirmativus, tum Quadraticis cum axe concursus H erit infra initium Abscissæ A, & proinde Area Methodo superiori inventa deficiet ab area quæsita AMC toto spatio $\frac{1}{2}$ FAq: sin valor quantitatis y fuerit negativus, tum H erit supra A: sin denique valor quantitatis y fuerit impossibilis, tum punctum H imaginarium est. Sic in primo

horum

horum trium Exemplo, ſi ponatur $x=o$ in æquatione Quadratricem definiente, *ſcil.*

$$\frac{4y^2+8ay+4a^2}{5}\times\sqrt{y+a}=x^2 \text{ ; erit, } \frac{4y^2+8ay+4aa}{5}\times\sqrt{y+a}=o,$$

Fig. 4. quæ reducta dabit $y=-a$. Sumptâ itaque AH, (ſupra A quia valor ejus eſt negativus) erit H punctum quæſitum.

Et in ſecundo Exemplo ubi $\frac{12y^2+4ay-8a^2}{15}\times\sqrt{y+a}=xx$, ſi po-

Fig. 5. natur $x=o$, erit $\frac{12y^2+4ay-8a^2}{15}\times\sqrt{y+a}=o$, quæ reducta dabit $y=+\frac{2}{3}a=AH$, cadente H infra A, quia y eſt valoris affirmativi.

In tertio Exemplo, ubi $\frac{2y^2+2a^2}{3}\times\sqrt{y^2+a^2}=xx$, poſito $x=o$,

Fig. 6. erit $\frac{2y^2+2a^2}{3}\times\sqrt{y^2+a^2}=o$, unde $y=\sqrt{-a^2}$; qui valor cum ſit impoſſibilis, concludendum eſt punctum H eſſe imaginarium, & proinde Quadratricem nuſquam cum Axe concurrere.

Ex dictis pariter manifeſtum eſt, AF eſſe ordinatam Quadratricis initio abſciſſæ convenientem, id eſt, ubi $y=o$. Et proinde ſi ponatur $y=o$, in æquatione Quadratricem definiente, hæc reducta dabit $\frac{1}{2}$ AFq ſpatium excedens vel deficiens ab Areæ Expreſſione Analytica Methodo generali invenienda. Illud proinde ab hac ſubſtractum, quando H cadit ſupra A, vel etiam quando H imaginarium eſt, dabit Quadraturam Areæ quæſitæ AMC, abſciſſæ AM competentis: Sin H cadit infra A, addendum eſt ſpatium $\frac{1}{2}$ FAq expreſſioni Areæ Methodo generali inveniendæ, ut inde habeatur Quadratura Areæ quæſitæ AMC.

Sic in primo trium Exemplo, ſi ponatur $y=o$,

Fig. 4. erit $\frac{4a^2}{5}\sqrt{a}=xx$; unde $\frac{2a^2}{5}\sqrt{a}=\frac{1}{2}xx$, & quia H hic cadit ſupra A, ideo

$$\frac{2y^2+4ay+2a^2}{5}\times\sqrt{y+a}-\frac{2a^2}{5}\sqrt{a}=AMC=\frac{1}{2}GMq-\frac{1}{2}FAq.$$

Et

Et in ſecundo Exemplo, ſi ponatur $y=o$, erit $\frac{2a^2}{15}\sqrt{a}=\frac{1}{2}x^2=\frac{1}{2}$ FAq, & quia H hic eſt infra A, ideo Fig. 5.

$$\frac{6y^2+2ay-4a^2}{15}\times\sqrt{y+a}+\frac{2a^2}{15}\sqrt{a}=AMC=\frac{1}{2}GMq+\frac{1}{2}FAq.$$

In tertio denique Exemplo, ponatur $y=o$, & erit $\frac{2a^3}{3}=xx$, unde $\frac{a^3}{3}=\frac{1}{2}xx=\frac{1}{2}$ FAq. Et qui H hic imaginarum eſt, ideo Fig. 6.

$$\frac{y^2+a^2}{3}\times\sqrt{y^2+a^2}-\frac{a^3}{3}=AMC=\frac{1}{2}GMq-\frac{1}{2}FAq.$$

Atque hoc modo procedendum eſt in omnibus aliis Quadraturis Analyticè expreſſis, ſive illæ particulares, ſive generales fuerint. Ita ut nunquam neceſſe ſit Qaadratricem Geometricè deſcribere, poſito enim (in æquatione Quadratricem definiente) $x=o$, habetur punctum H, & poſito rurſus $y=o$, habetur ſpatium $\frac{1}{2}$ FAq; ut oſtenſum eſt. Quodque in tribus ſuperioribus exemplis Quadratrices Geometricè deſcribere præſcripſerim, id ideo factum eſt, ut harum Regularum fontem aperirem.

DE
FIGURARUM
QUADRATURIS.

Pars Posterior.

DE Curvarum in certa genera, & gradus diſtinctione, pauca jam ſunt præmittenda. Curvas illas (cum Leibnitio) *Algebraicas* appello, quarum Naturæ exprimi poſſunt per æquationem, in qua duæ indeterminatæ quantitates, Lineas tantum rectas deſignant: Eſtque hoc primum Curvarum genus; quod ſub ſe infinitos Curvarum gradus continet, pro variis indeterminatarum dimenſionibus enumerandos.

Curvas illas (cum eodem Leibnitio) *Tranſcendentes* appello, quarum Naturæ exprimi poſſunt per æquationem, in qua, una ex indeterminatis lineam quandam Curvam deſignat. Et ſpeciatim, ſi hæc quantitas indeterminata Curvam deſignet Algebraicam ſeu primi generis, erit ipſa tranſcendens, Linea Curva ſecundi generis. Sin quantitas indeterminata æquationem ingrediens Curvam deſignet ſecundi generis, erit Tranſcendens (hac æquatione definita) Curva tertii generis; & ſic porro infinitum. Quodlibet etiam harum genus infinitos ſub ſe continet Curvarum gradus, pro quantitatis tranſcendentis Dimenſionibus enumerandos. Per Quantitatem Tranſcendentem ſemper intelligo quantitatem illam indeterminatam,

tam quæ lineam Curvam designat, quæque æquationem alterius Curvæ ingreditur.

Ex Corollario Lemmatis primi Part. I. patet Quadraturam cujuscunque Figuræ AMC dependere ab alia linea Curva FGH cujus intercepta PM sit æqualis applicatæ MC in Curva AC Figuram Quadrandam terminante. Deq; his notatu dignissimum est, quod si AC sint Curvæ primi generis tum Quadratrices FGH aliquando sunt Curvæ primi, aliquando etiam secundi generis; & si AC sint Curvæ secundi generis; tum Quadratrices FGH aliquando sunt Curvæ secundi, aliquando etiam tertii generis: & universaliter, cujuscunque generis sint Curvæ AC, tamen Quadratrices FGH aliquando sunt Curvæ ejusdem, aliquando proxime superioris generis. Figuræ verò AMC, quarum Quadraturas Methodo generali determinandas suscepi, à Curvis solummodo AC primi generis comprehenduntur, & proinde Quadratrix invenienda FGH aliquando erit Curva primi, aliquando Curva secundi generis. In parte hujus Tractatûs priori ostensum est, quomodo Quadratrix quælibet primi generis FGH, pro Quadranda qualibet eam admittente sit invenienda. Rem paulò difficiliorem jam aggredior, Quadratrices nimirum secundi generis FGH determinaturus, quando Quadranda AMC primi generis, aliam non admittit. Et spero me fundamenta tam generalia traditurum, ut eadem Methodus ad superiora Figurarum genera promoveri possit ab iis, qui majori fruuntur otio, quam quod præsens vitæ nostræ ratio largitur. Fig. 1.

Pro harum Quadraticum Transcendentium Tangentibus determinandis, necesse fuit novam mihi Methodum excogitare. Regulæ enim Leibnitii (quibus in superiori parte ubique usus sum) Cur• solummodo Algebraicas respiciunt; Ego saltem nihil inde Transcendentibus peculiare colligere potui, quod eodem jure ad aliorum Methodos non pertineat. Nemo tamen putet me à præstantissima ejus Methodo quidquam velle derogare; mihi enim persuasum est celeberrimum Virum, calculum suum differentialem, non modo ad hæc, sed multa alia recondita problemata posse porrigere. Ego interim Methodum meam, eodem ordine, quo mihi inter investigandum occurrebat, hic exhibeo.

Methodus Determinandi Tangentes Linearum Tranſcendentium.

Fig. 7. SIT AD Curva Tranſcendens cujuſcunque generis; AC Curva illa quæ Tranſcendentis ſpeciem determinat. Sitque Abſciſſa communis $AB=y$, Tranſcendentis ordinata $BD=x$, alterius Curvæ AC ordinata $BC=z$, quantitas Tranſcendens, ſeu portio Curvæ $AC=v$: Sintque Dd, Cc particulæ Curvarum AD, AC indefinitè parvæ; DE Tangens Curvæ AD, CF Tangens Curvæ AC, occurrentes Axi in punctis E, F; ducantur dc ad DC, & DH, CI ad AB parallelæ: cæteræ autem quantitates ſic notentur; incognita quæſita $EB=t$, $FB=b$, $FC=c$, $DH=Bb=CI=m$, $Hd=e$, $Cc=n$.

Lemma.

Quia $FC . FB :: Cc : Bb$. Hoc eſt, $c . b :: n . m$. Ideo erit $n=\frac{mc}{b}$.

Regula.

1. In æquatione Curvam AD definiente, pro y ponatur $y+m$, pro x ponatur $x+e$, & pro v ponatur $v+n$, hoc eſt, $v+\frac{mc}{b}$ (per Lem.) 2. Auferantur ex æquatione ſic compoſita omnes termini in quibus nec (m) nec (e) reperiuntur. 3. Auferantur omnes termini in quibus (m) vel (e) ſunt in ſeipſas, vel ſe invicem multiplicatæ. 4. In æquatione reliqua pro (m) ſubſtituatur ubique (t), & pro (e) ſubſtituatur x; unde æquatio ſecundum Algebræ regulas reducta dabit valorem Analyticum quantitatis t tangentem quæſitam DE determinantis.

Omitto Regulæ hujus demonſtrationem, quoniam deducitur ex generali omnium Methodorum fundamento, apud Geometras paſſim noto, & dilucidè à D. D. *Barrow* explicato.

Exemp.

Exemp. I. Sit $av + ay = xx$ æquatio definiens Curvam Tranſcendentem AD, cujus tangens ED quæritur. Sequendo partes Regulæ præcedentis, erit

$$1.\quad av + \frac{acm}{b} + ay + am = x^2 + 2xe + e^2.$$

$$2.\quad \frac{acm}{b} + am = 2xe + e^2.$$

$$3.\quad \frac{acm}{b} + am = 2xe.$$

$$4.\quad \frac{act}{b} + at = 2xx,$$

unde $t = \frac{2bxx}{ac + ab}$.

Quantitates b & c hic pro datis accipiuntur, quoniam AC eſt Curva inferioris generis, cujus tangens FC $= c$, & linea illam determinans FB $= b$ eodem modo ex Curva ſibi inferiori (& ſic porro donec tandem ad Curvam Algebraicam deveniatur) inveniri poſſunt, ut ſtatim oſtendam.

Manifeſtum eſt hoc modo infinitarum Tranſcendentium tangentes ſimul determinari; pro infinitis enim Curvis AC, infinitæ oriuntur Curvæ tranſcendentes AD, quarum omnium tangentes ex nuper invento quantitatis (t) determinantur. Ut ſi Curva AC ſit parabola communis cujus latus rectum ſit (a), & proinde $ay = z^2$, erit $b = 2y$, $c = \sqrt{4y^2 + ay}$; unde $t = \frac{4yxx}{a\sqrt{ay + 4y^2} + 2ay}$; & ſic ſubſtituendo particulares quantitatum b, c, valores, ex particulari Curvæ proprietate inveniendos, habentur particulares tranſcendentium tangentes.

Notandum eſt, non ſemper neceſſe eſſe omnem Calculum in præcedenti regula præſcriptum adhibere: ex eo enim Regulæ compendioſæ deduci poſſunt, prout factum eſt à Sluſio.

Ut

Ut ſi $av^r = x^s$, erit æquatio Regulæ quarta parte inventa, $\frac{tracv^{r-1}}{b} = sx^s$. Exinde etiam compendia formari poſſunt pro ſurdis & fractis, qualia ingenioſè excogitavit Leibnitius pro Curvis Algebraicis. Sed quia nullus erit horum in ſequentibus uſus, plura jam non addo.

Fig. 7. *Exemp.* 2. Sit $av + v^2 + \sqrt{y^4 + a^4} = xx$ æquatio exprimens Naturam Curvæ tranſcendentis AD, cujus tangens DE quæritur. Per compendium modo traditum, pro parte æquationis tranſcendente; & per Lebnitii regulas, pro parte ejus Algebraica erit,

$$\frac{act + 2cvt}{b} + \frac{2y^3t}{\sqrt{y^4 + a^4}} = 2xx.$$ quæ reducta dabit

$$t = \frac{2bxx\sqrt{y^4 + a^4}}{2by^3 + \overline{2cv + ac} \times \sqrt{y^4 + a^4}}.$$

Fig. 8. *Exemp.* 3. Sit jam Linea tranſcendens ADG Cyclois, cujus circulus genitor ACH, Axis AH, Baſis GH. Sitque D punctum in Cycloide datum, à quo ducenda eſt tangens DE. Ducta ordinatim applicata DB ſecans circulum in C; ſi que ut ſupra AB $= y$, BC $= z$, BD $= x$; ſitque circuli diameter AH $= 2a$. His poſitis, ex notiſſima Cycloidis proprietate erit $v + \sqrt{2ay - y^2} = x$ æquatio definiens Cycloidem datam, in qua quantitas tranſcendens AC $= v$. Itaque

$$\frac{ct}{b} + \frac{at - yt}{\sqrt{2ay - yy}} = x,$$ quæ reducta dabit

$$t = \frac{bx\sqrt{2ay - y^2}}{ba - by + c\sqrt{2ay - yy}} = \text{EB}.$$

Ob datum circulum ACH, dantur FC $= c$, FB $= b$, & proinde etiam quantitas t tangentem quæſitam DE determinans innoteſcit.

Fig. 9. *Exemp.* 4. Aſſumatur CD curva tertii generis cujus æquatio ſit $av = xx$, in qua x deſignat ordinatam BD, & v quantitatem tranſcendentem AC, (curvam ſcil. ſecundi generis); juxta Methodum noſtram erit

act

$\frac{act}{b} = 2xx$, unde $t = \frac{2\,bxx}{ac} =$ EB.

Ubi c designat ipsam tangentem quantitatis transcendentis, *scil.* FC, & b lineam inter ordinatam ejus BC & tangentis cum axe concursum F *scil.* FB: & quia datur Curva AC (hoc est æquatio illius naturam definiens) ideo dantur, c, b. Ut si $aw = uu$, in qua $u =$ BC, $w =$ AK. Erit $\frac{act}{b} = 2uu$, unde $t = \frac{2buu}{ac} =$ FB; dantur autem b, c in hac æquatione, quia datur Curva AK speciem transcendentis determinans; ut si $as = y^2$, in qua $y =$ AB, $s =$ BK, erit

TB $= b = \frac{1}{2}y$, TK $= c = \sqrt{\frac{1}{4}y^2 + \frac{y^4}{a^2}}$:

Diligenter enim notandum est c semper denotare ipsam tangentem, & b lineam inter ordinatam & tangentis cum axe concursum, in illa Curva quæ speciem istius Curvæ (cujus tangens quæritur) determinat. Sic in hoc Exemplo, quia AC est quantitas transcendens Curvæ AD cujus tangens ED quæritur; ideo in valore lineæ EB (per hanc Methodum invento) erit $b =$ FB, $c =$ FC; & quia AK determinat speciem Lineæ Curvæ AC, ideo in valore lineæ FB (per hanc Methodum inveniendo) erit TB $= b$, $c =$ TK; Adeo ut cujuscunque generis sit Curva AD, posset tamen semper invenire valor Analyticus lineæ EB, quæ tangentem quæsitum FD determinat; sub quo continentur tangentes omnium inferiorum generum; quæ omnes pro datis supponuntur, cum tangens cujuslibet Curvæ superioris, ex datis vel inventis tangentibus Curvæ generis proximè inferioris habeatur. Ut si quæratur tangens Curvæ sexti generis, quæ determinatur à Curva data quinti, quæ determinatur à Curva data quarti, quæ determinatur à Curva data tertii, quæ determinatur à Curva data secundi, quæ deniq; determinatur à Curva data primi generis: Per Vulgares Methodos invenietur tangens infimi seu primi generis; ex hac per Methodum nostram invenietur tangens Curvæ secundi, & ex tangente secundi invenietur tangens tertii, & ex tangente Curvæ tertii invenietur tangens Curvæ quarti, & ex tangente Curvæ quarti invenietur tangens Curvæ quinti, & sic denique ex tangente Curvæ hujus quinti invenietur tandem tangens quæsita Curvæ propositæ sexti generis: Calculus enim in omnibus idem est, nullaque eum quantitas ingreditur, nisi quæ datas harum Curvarum æquationes constituunt, & earum tangentes determinant. Nihil itaque jam deesse video, quod omnium Cur-

varum

varum transcendentium æque ac algebraicarum tangentes respicit, quodque in his, quæ jam explicui, continetur.

Methodus investigandi Quadratrices Transcendentes.

NOtandum est primò, quod duæ hic sint Lineæ Curvæ incognitæ, quarum altera est ipsa Quadratrix Transcendens, altera verò est Curva ipsius Transcendentis speciem determinans. Secundò, Quod Quadratrix semper sit Curva secundi generis, quia Figuras Curvis tantum primi generis tractandas assumimus.

Fig. 10, & 11. Sit AH, vel OH Curva Figuram Quadrandam ABH, vel ABHO comprehendens; AD ejus Quadratrix quæsita, & AC curva Quadratricis speciem definiens quarum communis abscissa $AB = y$ ordinatæ $BH = z$, $BD = x$, $BC = s$; quantitas transcendens $AC = v$, & (a) quantitas quælibet data & determinata unitatis locum supplens.

REGULA.

1. Æquationi per Problema primum Part. 1. inventæ addatur ev atque hæc eminenter continebit æquationem, quæ Quadratricem AD definiet; ubi e designat quantitatem incognitam sed determinatam: Facile enim demonstrari potest quantitatem v non ultra unam dimensionem ascendere, in Quadratrice transcendente cujuslibet Figuræ primi generis. 2. Valorem Ordinatæ z (omnibus ejus terminis sub vinculo involutis) multiplica per y, addantur omnes ejus potestates inferiores coefficientibus incognitis g, h, k, &c. affectæ; & summa omnium æquata quantitati s erit æquatio quæ eminenter continebit Curvam AC. 3. Per Methodum nostram Tangentium modo explicatam, ex æquatione Quadratricem AD eminenter continente inveniatur valor Analyticus Lineæ BL inter ejus perpendicularem AD & ordinatam DB interceptæ: 4. In hoc valore Analytico interceptæ BL substituantur valores quantitatum b, c, per communes Tangentium Methodos, ex æquatione Curvam AC eminenter continente inveniendos, ita ut nulla quantitas indeterminata præter y in valore interceptæ BL reperiatur. 5. Valorem lineæ BL sic inventus æquetur valori ordinatæ z: Termini hujus æquationis (à surdis & fractis liberatæ) ritè comparati coefficientes omnes incognitas determinabunt, quæ in propriis

locis

locis restitutæ dabunt æquationes quæ Curvas AC, AD definient.

Notandum, quod ideo necesse fuit æquationem, Curvam AC *includentem definire, quia aliter prorsus impossibile erit valorem interceptæ* BL *inventum cum valore ejus dato comparare, vel particularem Transcendentis Naturam determinare.*

PROB. I.

Circuli Quadraturam invenire.

SIT AHG semicirculus, cujus Diameter AG $= 2a$, unde $z = \sqrt{2ay - y^2}$, jam quia per Prob. 1. Part. 1, $ly + ma \times \sqrt{2ay - y^2} = x^2$ esset æquatio includens Quadratricem AD, siquidem illa esset Curva Algebraica seu primi generis; at cum talem pro Circulo nullam esse constet, ideo per Regulam præcedentem, 1. $ev + ly + ma \times \sqrt{2ay - y^2} = xx$ erit æquatio eminenter continens Quadratricem Circuli transcendentem AD. Fig. 10.

Et 2. $\sqrt{ka^4 + ba^3y + ga^2y^2 + pay^3 + qy^4} = s$ æquatio eminenter continens Curvam AC, quæ specialem Transcendentis naturam determinat. Sed quia calculum experto innotuit solos terminos, in quibus y^2, & y reperiuntur, eam comprehendere, ideo ut calculus simplicior fiat assumo $\sqrt{bay + gy^2} = s$. Quique (ad vitandas fractiones) facilior evadet si ponatur $\sqrt{2bay + gy^2} = s$. Plurima enim hujusmodi compendia inter operandum invenies. Atque hic semel monuisse sufficiat, me in sequentibus non omnes terminos Æquationum juxta præscriptum Regulæ, loco 1, & 2, assumendos adhibere, sed tot tantum eorum, quot per calculum vel aliunde, Curvas incognitas AD, AC eminenter continere cognosco.

3. Ex priori æquatione per Methodum tangentium jam explicatam invenio

$$BL = \frac{ec}{2b} + \frac{ma^2 + 3lay - may - 2ly^2}{2\sqrt{2ay - y^2}}.$$

Et ex posteriori æquatione invenio (4.) per communes tangentium Methodos

$$b = \frac{2bay + gy^2}{ba + gy},\ c = \sqrt{\frac{\overline{2bay + gy^2}|^2 + 2\,\overline{bay + gy^2} \times \overline{ba + gy}|^2}{\overline{ba + gy} \times \overline{ba + gy}}} = FC.$$

 Substi-

Subſtitutis itaque his valoribus quantitatum b, c, in nuper invento valore interceptæ BL, erit 5.

$$BL = \frac{e}{2}\sqrt{\frac{\overline{2bay + gyy + ba + gy}|^2}{2bay + gy^2}} + \frac{ma^2 + 3lay - may - 2ly^2}{2\sqrt{2ay - y^2}} =$$

$$= \sqrt{2ay - y^2}.$$

Hæc æquatio à fractis & ſurdis liberata dabit hanc,

$$\left.\begin{array}{l} +4llg \\ -8lg \\ +4g \end{array}\right\} y^6 \left.\begin{array}{l} -12llg \\ +28lg \\ +4lmg \\ -16g \\ -4gm \\ +8llb \\ -16lb \\ +8b \end{array}\right\} ay^5 \left.\begin{array}{l} -10lmg \\ +12mg \\ +9llg \\ -24lg \\ +16g \\ +m^2g \\ -24llb \\ +56lb \\ +8lmb \\ -32b \\ -8mb \end{array}\right\} a^2y^4 \left.\begin{array}{l} +6lmg \\ -8mg \\ -2m^2g \\ -20lmb \\ +24mb \\ +18llb \\ -48lb \\ +32b \\ +2m^2b \end{array}\right\} a^3y^3$$

$$\left.\begin{array}{l} +m^2g \\ +12lmb \\ -16mb \\ -4m^2b \end{array}\right\} a^4y^2 + 2m^2ba^5y = \left.\begin{array}{l} -e^2g^2 \\ -e^2g \end{array}\right\} y^4 \left.\begin{array}{l} +2e^2g^2a \\ +2e^2ga \\ -2e^2bga \\ -2e^2ba \end{array}\right\} y^3$$

$$\left.\begin{array}{l} +4e^2bga^2 \\ +4e^2ba^2 \\ -e^2b^2a^2 \end{array}\right\} y^2 + 2e^2b^2a^3y.$$

Facta comparatione terminorum hujus æquationis, erit prima $4llg - 8lg + 4g = 0$, unde $l = 1$. Secunda comparatio erit $-12llg + 28lg + 4lmg - 16g - 4mg + 8llb - 16lb + 8b = 0$, ſi ſubſtituatur valor quantitatis l, invenietur omnes terminos ſe mutuò deſtruere, unde nullius coefficientis determinatio ex hac ſecunda comparatione habetur. Tertia erit $-10lmg + 12mg + 9llg - 24lg + 16g + m^2g - 24llb + 56lb + 8lmb + 8mb - 32b \times aa = -g^2 - g \times e^2$, ſubſtituto valore quantitatis l, & ablatis quæ ſe mutuò deſtruunt, erit

$$m^2g + 2mg + g \times a^2 = -g^2 - g \times e^2, \text{ unde } g = \frac{\overline{-m^2 - 2m - 1} \times a^2 - e^2}{e^2};$$

Quarta,

Quarta, $6lmg + 8mg - 2m^3g - 20lmb + 24mb + 18llb - 48bl + 32b + + 2m^2b \times a^3 = \overline{2g^2 + 2g - 2bg - 2b} \times ae^2$, quæ ritè tractata dabit $g = \frac{\overline{-m^2 - m} \times \overline{a^2 - e^2}}{e^2}$; & proinde harum Fractionum numeratores sunt æquales *scil.* $-m^2a^2 - 2ma^2 - a^2 - e^2 = - m^2a^2 - ma^2 - e^2$, unde $m = -1$; subst ituto itaque valore quantitatis (m) invenietur $g = \frac{-e^2}{e^2} = -1$. Quinta, $a^4 \times m^2g + 12lmb - 16mb - 4m^2b = \overline{4bg + + 4b - bb} \times e^2a^2$, unde (substitutis g, l, m repertis) erit tandem $eb = a$. Sexta denique $2m^2ba^5 = 2b^2e^2a^3$, unde $be = \frac{a^2}{e}$; ergo $a = \frac{a^2}{e}$, unde $e = a$, & proinde $b = \frac{a}{e} = i$; & sic tandem omnes coefficientes incognitæ inveniuntur *scil.* $l = b = 1$, $m = g = -1$, & $e = a$. Hi valores in propriis æquationibus substituti dabunt $av + \overline{+ y - a} \times \sqrt{2ay - y^2} = xx$ æquatio definiens Circuli Quadratricem AD, & $\sqrt{2ay - y^2} = s$ æquatio definiens Curvam AC; & quia $z = \sqrt{2ay - y^2}$, ideo $z = s$, ideoque non differunt AC, AH, nec specie nec magnitudine: unde constat Lineam Curvam quæ Quadratricem Circuli determinat, esse ipsius Circuli circumferentiam. Et juxta Lem. 1, Part. 1.

$$\frac{av + \overline{y - r} \times \sqrt{2ay - y^2}}{2} = \tfrac{1}{2}xx = \text{ABH}.$$

Ubi $v = \text{AH}$ (nam AC, AH hic coincidunt ut dictum) atque hæc est vera Circuli Quadratura Transcendens quæsita, nec possibile est illam aliter per æquationem finitam exprimere.

Prob. II.

Hyperbolæ Quadraturam invenire.

SIT OH Hyperbola æquilatera, cujus centrum A, vertex O, semiaxis $\text{AO} = a$, abscissa $\text{AB} = y$, Ordinata $\text{BH} = z$, unde $z = \sqrt{y^2 + aa}$ per Prob. 1, Part 1. $ly + ma \times \sqrt{y^2 + a^2} = xx$ esset æquatio definiens Quadratricem AD, si quidem illa esset Curva primi generis, sed quia nullam esse talem pro Hyperbola constat; ideo per Regulam præcedentem 1. $ev + ly + ma \times \sqrt{y^2 + a^2} = xx$ eminenter Fig. 11.

continebit Quadratricem Hyperbolæ Tranſcendentem. Et 2. erit $\sqrt{k + fy^2 + gy^2 + py^3 + by^4} = s$ æquatio eminenter continens Curvam AC$=v$; ſed calculum expertus inveni priorem ſub $ev = x^2$ poſteriorem vero ſub $s = \sqrt{k + by^4}$ comprehendi, & ideo has pro illis uſurpo. 3. Per Methodum noſtram Tangentium invenio interceptam BL$= \frac{ec}{2b}$. Et 4. per communes Methodos invenio etiam

$$b = \frac{k + by^4}{2by^3} = \text{FB}, \quad c = \frac{\sqrt{\overline{k + by^4}|^2 + \overline{k + by^4} \times 4bby^6}}{2by^3} = \sqrt{bb + ss} = \text{FC}.$$

Unde 5. $\text{BL} = \frac{ec}{2b} = \frac{e}{2}\sqrt{\frac{k + by^4 + 4bby^6}{k + by^4}} = \sqrt{y^2 + a^2} = \text{BC}.$

Hæc æquatio à fractis & ſurdis liberata dabit,

$$+4bbeey^6 + be^2y^4 + ke^2 = 4by^6 + 4ba^2y^4 + 4ky^2 + 4ka^2;$$

prima comparatio $4bbee = 4b$, unde $b = \frac{1}{e^2}$; ſecunda erit hæc $bee = 4baa$, unde $e = 2a$; & proinde $b = \frac{1}{ee} = \frac{1}{4aa}$; tertia $ke^2 = 4ka^2$, unde rurſus $e = 2a$; quarta denique $4ky^2 = o$, unde $k = o$; & ſi omnes terminos præcedentium æquationum in hoc calculo retinuiſſem, inveniſſem pariter (ſed prolixiori calculo) $l = o$, $m = o$, $f = o$, $g = o$, $p = o$. unde conſtat ſolos terminos coefficientibus e & b affectos prædictas æquationes conſtituere, erit itaque $2\,av = xx$, æquatio definiens Quadratricem Hyperbolæ Tranſcendentem AD, & $s = \sqrt{\frac{1}{4aa}y^4}$, hoc eſt $2\,as = y^2$ equatio definiens Curvam AC$=v$, quæ in hoc caſu eſt Parabola, cujus latus rectum eſt $2\,a$. Atqui per Lem. 1, Part. 1.

$$av = \tfrac{1}{2}xx = \text{ABHO}.$$

Quæ eſt vera Hyperbolæ Quadratura Tranſcendens, quæque à longitudine Lineæ Parabolicæ AC dependet, ut jam antea ab Heuratio notatum fuit in Epiſtola ſua ad Carteſii Geometriam annexâ.

Prob. III.

Fig. 11. INvenire Quadraturam Figuræ ABHO, cujus proprietas eſt $z = \sqrt{y^3 + a^2}$. Per Prob. 1. Part. 1. $ly + ma \times \sqrt{y^3 + a^2} = xx$ eſſet æquatio definiens Quadratricem AD, ſi modo illa eſſet Curva Algebraica; at quia hæc Figura talem non admittit, ideo per Regulam præcedentem erit 1. $ev + ly + ma \times \sqrt{y^3 + a^2} = xx$, in qua v denotat

tat portionem Curvæ AC, cujus æquatio eminenter continetur ſub hac $\sqrt{p+qy+dy^2+ky^3+gy^4+hy^5}=s$, calculum ſequutus inveni $l=o$, $m=o$, $p=q=d=k=g=o$. Ideoque $ev=xx$ erit æquatio definiens Quadratricem tranſcendentem AD; & $\sqrt{hy^5}=s$ æquatio definiens Curvam AC, quæ Tranſcendentis ſpeciem determinat. 3. Per Methodum Tangentium jam explicatam invenio interceptam

BL$=\frac{ec}{2b}$: & per comunes Methodos invenio 4, $b=\frac{2y}{5}$, unde

$c=\sqrt{bb+ss}=\sqrt{\frac{4y^2}{25}+hy^5}$; erit ergo 5.

BL$=\frac{ec}{2b}=\frac{e}{4}\sqrt{4+25hy^3}=\sqrt{aa+y^2}$. Hæc à fractis & ſurdis liberata $4ee+25he^2y^3=16aa+16yyy$; prima comparatio $4e^2=16a^2$, unde $e=2a$, ſecunda $25he^2=16$, unde $h=\frac{4}{25aa}$; Ideoq; $2av=xx$ definiet Quadratricem AD, & $s=\sqrt{\frac{4y^5}{25a^2}}$ definiet Curvam AC$=v$, unde per Lem. 1. Part 1: $av=\frac{1}{2}xx$. Dependet itaque propoſitæ Figuræ Quadratura ex longitudine lineæ Curvæ AC cujus proprietas eſt $s=\frac{2}{5a}\sqrt{y^5}$.

Prob. IV.

INvenire Quadraturam Figuræ ABHO, cujus Natura ſit $z=\sqrt{y^4+a^2}$; Fig. 11. per Prob. 1, Part. 1. $ly+ma\times\sqrt{y^4+a^2}=xx$ eſſet æquatio definiens Quadratricem AD, ſi illa Algebraica fuiſſet; at quia Tranſcendens eſt, ideo $ev+ly+ma\times\sqrt{y^4+a^2}=xx$, per primam præcedentis Regulæ partem. Et 2. $s=\sqrt{hy^6+gy^5+py^4}$, &c. æquatio eminenter continens Curvam AC, à qua Tranſcendens Quadratrix AD determinatur. Inveni tamen $l=m=g=p$, & $c=o$; ideoque $ev=x^2$ Curvam AD, & $s=hy^6$ Curvam AC definiet. 3. Ex priori per Methodum præcedentem invenietur

BL $=\frac{ec}{2b}$. 4. Per communes Tangentium Methodos invenietur.

$c=\frac{y}{3}\sqrt{1+9hy^4}=$FC, $b=\frac{y}{3}=$FB; ſubſtitutis itaque his valoribus quantitatum b, c, in nuper invento valore interceptæ BL, erit.

BL$=$

$BL = \frac{ee}{2b} = \frac{1}{2}\sqrt{ee + 9beey^4} = \sqrt{a^2 + y^4}$.

Hæc à ſurdis & fractis liberata $ee + 9beey^4 = 4aa + 4y^4$.

Prima comparatio erit $e^2 = 4a^2$, unde $e = 2a$. Secunda erit $9bee = 4$, unde $b = \frac{1}{9aa}$. Et proinde $2av = xx$ Quadratricem AD; & $s = \sqrt{\frac{y^6}{9a^2}}$, hoc eſt, $s = \frac{y^3}{3a}$ eſt æquatio Curvam AC definiens. Et per Lem. 1, Part. 1. $av = \frac{1}{2}x^2 =$ ABHO, ubi v deſignat portionem Curvæ AC jam inventæ.

Atque jam Methodum hanc me ſufficienter explicaſſe credo, ex qua multa præclara Theoremata pro Quadratricibus Tranſcendentibus deduci poſſunt, ope Lemmatis 2, Part. 1. qualia exinde pro Quadratricibus Algebraicis deduxi. Habes itaque, benigne Lector, quæ de Figurarum Quadraturis, hactenus meditatus ſum; in quibus, ſi aliquid ad Geometriam promovendam reperias, me tempus & operam non inutiliter collocâſſe judicabo.

RESPON-

RESPONSIO AD LITERAS Domini D. T. *Lipsiam* Missas

FEB. 20. 1686.

SExtus jam labitur Annus, ex quo hæ Literæ ad manus meas pervenerunt; clarissimum Autorem tamen bile perquam fervida illas scripsisse percipiens, credebam me, ex officio humanitatis, Responsionem meam differre debuisse, ut ille interim, *Mentis Medicinæ* dosi sæpius repetita, Animum suum tandem ab Ira purgaret, facilius etiam errores percipiat, quos ab ipso in Geometria commissos jam sum ostensurus; atque hoc pacto *Medicum* simul & *Geometram* meliorem se præstititisse demonstrabit. Nunc itaque breve huic Epistolæ Responsum dabo, neglectis consultò iis, quæ in præstantissimos hujus ævi Geometras, eosque immeritò, effudit; & quæ in meipsum opprobria congessit. Sciat etenim scurvilem, quo utitur, Scribendi stylum, cum sit hominibus ingenuæ educationis indignus, etiam à moribus nostris esse quam maximè alienum.

Acta eruditorum Lipsiæ publicata mense *Octobri*, 1683, nonnulla

nulla cogitata circa Figurarum Quadraturas propoſuere, quibus Autor *D. T.* Methodi cujuſdam Specimen contineri aſſerit. Eadem Acta Menſe *Maio* 1684, aliud ſcriptum exhibuere, cujus Autor Anonymus, poſtquam de prædicta Methodo tanquam à ſe inventa multùm præfatus fuerat, concludebat tandem eam ſibi non uſquequaque arridere. Ego, cum Methodum iſtam penitius inſpexiſſem, non putabam rem tantæ molis, ut Geometris uno pluribus digna haberetur: & quia nihil tum extabat, quod contrarium ſuaderet, credebam me quæcunque de hac re in ſcedulis citatis ſcripta erant, tanquam ab uno aliquo viro ſcripta ſupponere tutò potuiſſe: præſertim cum nihil contra iſtam Methodum afferebam, quod non æquè valeret, ſive unum ſive plures haberit Autores. Satis tamen mirari nequeo Dominum *D. T.* dicere potuiſſe Schædam Menſe *Maio* ſcriptam Literas *G. G. L.* titulo præfixas habuiſſe, ex quibus non ipſum, ſed Dominum Leibnitium ſchedulæ iſtius Autorem eſſe cognoſcerem; quod falſiſſimum eſſe, cuivis vel leviter eam inſpicienti conſtat. Sed Reſponſione non opus eſt ad talia, quæ *D. T.* tantopere cavillatur, quæque cum verbis Domini *G. G. L.* collata oſtendunt non ipſum, ſed Dominum Leibnitium Methodi iſtius legitimum eſſe Autorem: idque jam manifeſtum fecit Dominus *G. G. L.* in ſchedula Actorum cui titulus (*De Geometria recondita*, &c.) in qua expreſsè ſe ejus Autorem eſſe aſſerit, " *& eandem Domino* D. T. *decem ab hinc Annis communicâſſe, cum Pariſiis de rebus Geometricis crebrerrimè loquerentur*, quo tempore *D. T.* per alias planè vias incedebat, " *dum interim ipſi* (Leibnitio) *hæc Methodus erat familiariſſima*. Quæque itaque in hoc negotio contra me tam iratus ſcripſit *D. T.* omnia huc redeunt, quod ipſum Plagiarium eſſe tum neſcirem, & quod Colloquia, quæ ſexdecim ab hinc annis ipſi cum Domino *G. G. L.* intercedebant, divinare non potuerim.

Sed ad ipſam Methodum rurſus conſiderandum accedo, prout illa à Domino *D. T.* explicatur; nullus enim dubito quin celeberrimus Leibnitius eam perfectiſſimè intelligat. Illa autem his tribus continetur. 1. Data Quadratrice invenire Quadrandam, vel, quod idem eſt, invenire Curvam cujus Area per datam quamlibet æquationem exprimatur: hoc autem ſolius Domini *Barrow* inventum eſt, in *pag.* 125. *Lect. Geom.* 2. Invenire, certâ Methodo, Quadrandam generalem ſimpliciſſimam, quæ datam quamlibet Quadrandam particularem eminenter contineat. 3. Quadrandæ generalis ſic inventæ terminos, cum reſpectivis terminis Æquationis Curvam propoſitam exprimentis comparare, ut inde habeatur Quadratrix quæſita

quæſita, vel pateat Quadraturæ impoſſibilitas. Hoc verò Carteſii inventum eſt, qui in ſecundo Geometriæ ſuæ libro Methodum expoſuit ſolvendi Problemata per æquationum comparationem, quam per Tangentium inventionem egregiè illuſtravit, & expreſsè infinitis aliis Problematis inſervire poſſe aſſerit. Nihil itaque ſupereſt, quod Domino *D. T.* tribuatur, niſi ſecundum ſolvat Problema, Regulam certam exhibendo, qua debitum Theorema eligatur. Quam infœlices hucuſque fuerint omnes ejus conatus ex ſequentibus luculenter apparebit.

Vid. *pag.* 49. *Geom. Edit. Amſtel.* 1683.

Notandum eſt *D. T.* duas Regulas tradidiſſe, quarum prior continetur in ſpecimine, & ex dimenſionibus quantitatis *z* Theorema eligendum jubet, abſque ullo ad dimenſiones quantitatis *x* reſpectu, ut ex ipſius verbis liquido conſtabit, " *Quia ordinatim applicata ad duas dimenſiones aſcendit, ſecundum Theorema eligendum* " *(ſi tres habuiſſet dimenſiones tertium Theorema fuiſſet eligendum, &* " *ſic porro).* Regulam generalem eſſe, poſtrema ejus verba (*& ſic porro*) apertè indicant. Et cum ne verbum quidem amplius addiderit, quis non videt illum voluiſſe, ex ſolius ordinatim applicatæ *z* dimenſionibus, debitum Theorema eligere? Cum ne minimam dimenſionum abſciſſæ *x* in tota ſua Regula mentionem fecerit. Sed ex Animadverſione mea percipiens infinitas eſſe Figuras, quæ ad Methodum ſuam ſic explicatam reduci nequeunt, ad ſecundam Regulam confugit, quam in his literis edidit, dicens in electione Theorematis non tantum ordinatæ *z*, ſed etiam abſciſſæ *x* dimenſiones eſſe reſpiciendas. Et ne videatur novam Regulam exhibuiſſe, prioris Regulæ verba peſſimè mutavit, ubi ait ——— " *Dixeram in* " *ſpecimine*, quando *ordinatim applicata*, &c. Cum re vera in Specimine dixiſſet, " quia *ordinatim applicata*, &c. Illa quidem laxiorem ſenſum admittunt; hæc verò tam abſolutè ordinatam reſpiciunt, ut omnem Abſciſſæ conſiderationem prorſus excludant: quod ideo notari velim, ut pateat quam miſerè in his Literis tergiverſetur. Dein procedit, " *Hoc Autor hic adeo abſolutè intellexit, ac ſi* " *nullus* juxta me reſpectus habendus eſſet ad Regulas comparatio- " num, *quas tamen expreſsè dixi adhibendas*. An nimis abſolutè ipſum intellexerim, verborum ſuorum mutatio ab ipſo facta teſtatur. Regulas autem comparationum quod attinet; ego quidem nihil de iis in Animadverſione mea adduxi, ipſum enim ex Carteſii Geometria edoctum eas ritè adhibuiſſe percipiebam. Sed cum Regulæ comparationum, æquationes comparandas aliunde datas vel inventas ſupponunt, abſurdum eſt Theorematis electionem ex iis deducendam eſſe aſſerere; quod etiam ipſius Domini *D. T.* proceſſus indicat:

 ut

ut videre est in *pag.* 435 & 436, *Actorum Anni* 1683; ubi, assumptâ Figurâ, cujus ordinata est duarum dimensionum, juxta Regulam § 1. traditam, secundum Theorema eligit, nullam aliam electionis rationem adducens, quam *quia* ordinatim applicata *z* ad duas dimensiones assurgit. Tum § 2, In Theoremate sic electo quantitates B, C, *&c.* restituere jubet. Dein in § 3, fiat (inquit) comparatio omnium horum terminorum Theorematis hujus, *&c.* comparatio itaque juxta ipsum instituenda est cum Theoremate aliunde electo, nimirum ex dimensionibus ordinatæ *z* juxta præscriptum Regulæ § 1. exhibitæ, adeoque non verba tantum istius Regulæ, sed etiam Domini *D.T.* ejusdem applicatio luculenter confirmat nullum respectum habendum esse quantitatis *x* in eligendo, sed tractando Theoremate ex solius ordinatæ *z* dimensionibus electo. Hunc Regulæ suæ defectum in Animadversione mea demonstratum se excusare posse sperat, innuendo sub initio hujus Epistolæ non integram Methodum, sed illius tantum *Specimen* aliquod in actis Eruditorum contineri. In speciminibus quidem solent res compendiosiùs tractari; at partem rei tractandæ dimidiam, eamque primariam prorsus intactam relinquere, *Speciminis* nomen non patitur. Hæc saltem excusatio nullo jure ad Dominum *D. T.* attinere potest, qui tot *Speciminis* sui lineas inani jactantia repleverit, dum Veterum & Recentiorum inventa longissimè prætervexisse gloriatur. Rem sane Modestiæ & Speciminis brevitati magis convenientem, multóque magis Geometris gratam præstitisset, si *has Laudes, quas nemo sanus sibi ipsi tribueret*, in aliud tempus reservasset, & interea Regulam certam eligendi debitum Theorema exhibuisset. Atque hæc de vulgaribus istis, & Domino *D. T.* nimis familiaribus, errorum Asylis dicta sufficiant. Ad posteriorem ejus Regulam jam transeundum est, quam iisdem, quibus prior, erroribus involutam reperiemus.

Vid. *Acta Erud.* pag. 370. *An.* 1686. & pag. 437. *An.* 1683.

" *Esto* (inquit) $zz = \frac{x^{10}}{a^8}$, *cum præceperim si* z *sit duarum dimensio-*
" *num secundum meum Theorema esse eligendum (intellige si Regulæ*
" *comparationum hoc patiantur) & vero, quia hic habetur* x^{10}, *quod*
" *in dicto Theoremate non adest, clarè pateat comparationem cum memo-*
" *rato Theoremate institui non posse; hoc in casu tertium Theorema assumi*
" *debere concluderem, in quo necessariò aliqui termini erunt, qui æqua-*
" *les dimensiones cum quantitatibus* z *&* x *obtineant.*

Propositæ autem Figuræ Quadratrix est $6a^4y = x^5$, atque hæc nec in tertia, nec in quarta, sed in quinta (ad minimum) Quadratrice

trice generali continetur; & proinde in hoc casu, non tertium, sed quintum Theorema assumendum erat. Falsa itaque est ipsius Regula, quæ tertium assumere jubet, quando non nisi quintum Theorema assumendum est. Et si tam gravem commiserit errorem in Figura tam simplici, ab ipso etiam ad novam suam Regulam examinandam proposita; quam exigui sit momenti in aliis Figuris magis compositis, ipsi Domino *D. T.* & aliis abundè patet. Ad pleniorem verò confirmationem tres alias Figuras hìc ascribere visum est, $zz = \frac{x^3 + axx}{a}$; $z^2 = \frac{x^4 + m^2x^2}{pp}$; $z = \frac{x^5 + ax^4}{aaa}$; Quia in his z non ultra duas, nec x ultra decem dimensiones assurgit, ideo juxta Dominum *D. T.* nulla harum ultra tertium ejus Theorema ascendit; & tamen ex earum Quadraturis in Methodo nostra determinatas constat primam cum quinto, secundam cum sexto, & tertiam cum septimo ejus Theoremate comparandam esse. Et de secunda harum trium observandum est; illam jam in Animadversione mea propositam fuisse; & licet primaria esset Figura, quam contra ejus Methodum adduxi, nulla tamen illius in his literis facta est mentio, quam absque dubio fecisset, si quovis modo illam ad regulas suas revocare potuisset: hoc enim præ cæteris, quas illi tum proposueram, peculiare habet, quod cum secundo ejus Theoremate comparanda sit, sive priorem, sive Regulam sequamur posteriorem. Habes itaque, benigne Lector, brevem, plenam tamen & perspicuam demonstrationem Errorum, quos monitus etiam commisit Dominus *D. T.* in tantopere jactata Quadraturas determinandi Methodo. Consultiùs esset, hæc celeberrimo eorum Autori *G. G. L.* tractanda committere, & viis suis antiquis incedere, quibus insistebat priusquam ipsum Parisiis conveniret: Viatoribus enim per vias incognitas incedentibus aberrare sæpissimè contingit.

In conclusione hujus Epistolæ totus est (ut solet) in semetipsum & Inventa sua prædicando occupatus. *Ille* (si credas) *Series habet* Newtoni *seriebus simpliciores, magisque genuinas: Ejus Theoremata* Barovianis *multò prævalent: Ille specimen exhibuit Methodi Tangentium Universalis, qualem nemo adhuc publicavit: Omnes Curvas conceptibiles formare novit, quod nec à Cartesio, nec ullo alio publicatum: Ille tribus lineis prolixas Ja. Gregorii nostratis demonstrationes explicare potest: tantum tribus quatuorve præstare potest, quantum Dominus* Barrow *magno præstitit Theorematum numero: Ille Problema illustri Hugeniano Problemati simile proposuit.* Aliorum quærere laudes, apud omnes meritò habetur inhonestum; at tot & tam immeritas, ex ali-

Vid. *Act. Erud. An.* 1686. p. 175, 176.

orum famæ ruina petitas in se cumulare laudes, quis non abhorreat? Ego præstantissimorum Virorum, quos etiam non lacessitus aggreditur *D. T.* Causam sibi ipsis vel aliis suscipiendam committens pauca addam in defensionem celeberrimi Viri D. D. *Barrow*, in quem, mea de causa, tam furiosè invehitur.

Inventa Domini Barrow *nimium quantum extollit.*—— Piæ memoriæ Virum, in omni scientiarum genere versatissimum, ejusque egregia inventa satis extollere non potui: tantum enim & meritò apud omnes probos & eruditos honorem assequutus est, ob summam Virtutem, & Naturæ suæ suavitatem, profundæ & omnigenæ eruditioni conjunctam. Iis, quæ ad famam ejus minuendam notavit *D. T.* hæc repono. Primò, generalia illa Theoremata, quæ inventis Domini *Barrow* prævalere ait, non sua, sed ipsius esse D. *Barrow* inventa: Sunt enim Exempla tantùm problematis ejus universalis ab ipso in pag. 125, *Lect. Geom. soluti*; atque hæc vera causa est, ob quam Dominus *D. T.* Theorematum inventionum celaverit; quæ tamen quivis Tyro ope Problematis Baroviani invenire potest. Secundò, Dominus *D. T.* Methodum suam pro *maximis* & *minimis* ex Lectionibus Geometricis Domini *Barrow* desumpsit, ut primò intuitu constabit cuivis, qui conferet pag. 146. *Lect. Geom.* cum pag. 122. *Act. Erud. Anni* 1683. Ubi eandem prorsus Methodum reperiet, primò à Domino *Barrow* inventam, & postea à Domino *D. T.* sibi ipsi arrogatam. Tertiò, non Heuratio, sed Cavallerio, quem strenuè contra Tacquetum defendit D. *Barrow*, debetur hæc Methodus inveniendi Theoremata: Ante Heuratium innotuit Triangula similia esse proportionalia, & Curvas esse Polygona indefinitorum laterum, quæ ad hoc efficiendum (ipso domino *D. T.* fatente) cognovisse sufficit. Mirari itaque desinat D. *Barrow*, Heuratii mentionem nullibi fecisse; & miretur potius, quod ipse tam ignominiosè de illo scripserit, ex cujus Operibus expilavit, quicquid solidi de Figurarum Quadraturis, quicquid etiam de *maximis* & *minimis* hactenus ediderit: Nullus equidem est, quin jure mirari possit, Dominum *D. T.* sua toties decantata Theoremata, illius inventis præferenda esse asserere, cum unius tantum horum Problematis pars, eaque perexigua, illa omnia existant. Quod fusius demonstrarem, nisi scirem, quod, quicunque hæc nostra lecturus sit, comparando loca citata rem ita se habere percipiet. Quartò, per Methodum meam ope Theorematis Barovinani sic solvitur Problema, cujus solutionem extemplo sine calculo se dedisse ait.

Fig. 12. Sit EDK Curva Geometrica, cujus Tangens AD, Ordinata DB$=x$,

DB=x, Abſciſſa EB=y, ſitque DC Tangenti AD perpendicularis; Curvam EDK determinare, ubi Quadratum BC in lineam BD ſemper æquale ſit Cubo datæ Lineæ FG=a. Ex natura Problematis erit $\overline{BC}q \times x = a^3$, unde $BC = \sqrt{\frac{aaa}{x}}$; & per Methodum noſtram præcedentem, $ny\sqrt{\frac{a^3}{x}} = x^2$, & determinando ($n$) ut jam explicavi invenies $n = \frac{5}{2}$, unde $\frac{5}{2}y\sqrt{\frac{aaa}{x}} = xx$, ſeu $25\, a^3 y^2 = 4x^5$, quæ æquatio definit naturam Curvæ quæſitæ EDK; erráſſe proinde Typographum video, qui in Leibnitii ſolutione poſuit $4\, a^3 y^2 = 25\, x^5$.

Nec majori difficultate eodem modo invenietur Solutio hujus univerſalis; *Invenire Curvam EDK, ubi quæcunque poteſtates linearum BC, BD inter ſe multiplicatæ, æquales ſint convenienti poteſtati lineæ datæ* FG=a; Sit e exponens lineæ BD, & r exponens poteſtatis lineæ BC, unde $r+e$ eſt conveniens exponens lineæ datæ, FG=a; ideoque ex natura problematis $\overline{BC}|^r \times x^e = a^{r+e}$; unde $BC = \sqrt[r]{\frac{a^{r+e}}{x^e}}$; ideoque ad problema meum generale reducitur, juxta cujus ſolutionem erit $ny\sqrt[r]{\frac{a^{r+e}}{x^e}} = xx$; & determinando ($n$) ut jam expoſui invenies utique $n = \frac{2r+e}{r}$; ideoque Æquatio erit $\frac{2r+e}{r}y\sqrt[r]{\frac{a^{r+e}}{x^e}} = xx$. Atque ſic non unum tantum, eumque ſimpliciſſimum, ſed omnes poſſibiles caſus ſub uno calculo complexus ſum.

Hæc ſunt, quæ à me invito extorſit Dominus *D.T.* quæque ſpero finem mihi propoſitum habitura, qui quidem alius non eſt, quam ut ipſe de ſuo penu meliora promat, vel ſaltem de aliorum inventis humanius ſcribere diſcat.

NOVA

NOVA METHODUS

Determinandi

LOCA GEOMETRICA.

OMnes Locorum Solidorum Casus ad quatuor Theoremata generalia reduco, quorum primum, omnes Casus in quibus Locus quæsitus est *Parabola*; secundum & tertium omnes Casus, in quibus locus quæsitus est *Hyperbola*; quartum denique omnes Casus, in quibus Locus quæsitus est *Ellipsis*, universaliter comprehendit; atque has peculiares habent utilitates; quod nullas Æquationis primò conceptæ reductiones vel transmutationes requirant; Linearum in quolibet casu ducendarum positiones simul & magnitudines definiant, absque ullo respectu ad multiplices illas regulas pro variis signis + & −, & æquationum variis formulis considerandas.

THEOR. I.

Fig. 13. *SINT* x, y, *quantitates incognitæ & indeterminatæ, & fiat alterutrius harum, puta* x, *initium certum & immutabile punctum* A; *& ab hoc puncto per rectam* AE *positione datam indefinitè se extendere intelligatur; sitque Angulus datus vel assumptus, quem faciunt* x *&* y, *æqualis Angulo* AED.

CONSTRUCTIO.

Ducantur AK, BC parallelæ ad ED, per A & C ducatur linea indefinita ACF, cui à puncto K parallela agatur alia Linea indefinita KH, in qua assumatur punctum aliquod G; tum vertice G circa Diametrum GH describatur Parabola GD: quantitates autem

autem sic notentur, AE$=x$, ED$=y$, AB$=m$, BC$=n$, AC$=e$, AK$=k$, KG$=l$; sitque r latus rectum Parabolæ, ex cujus notissima proprietate invenietur.

THEOR. I. PARS I.

$$\left.\begin{array}{l} y^2+\frac{2nxy}{m}-2ky+\frac{nnxx}{mm}-\frac{2nkx}{m}+kk \\ \qquad\qquad\qquad\qquad -\frac{rex}{m}+rl \end{array}\right\}=0.$$

Secundò, *Sit* A *initium immutabile quantitatis* x *per rectam* AE *positione datam extensæ ; Sitque Angulus quem faciunt* x *&* y *æqualis Angulo dato vel assumpto* AED. Fig. 14.

CONSTRUCTIO.

Per A ducatur AL parallela ad ED, & à puncto ejus aliquo B agatur BC parallela ad EA, per A, C ducatur indefinita ACF; tum à puncto aliquo K in linea AE sumpto ducatur KH parallela ad AF; denique à puncto G in linea KH sumpto circa Diametrum GH describatur *Parabola* GD; quantitates etiam, ut supra, notentur, AE$=x$, ED$=y$, AB$=m$, BC$=n$, AC$=e$, AK$=k$, KG$=l$, latus rectum $=r$; erit rursus.

THEOR. I. PARS 2.

$$\left.\begin{array}{l} x^2+\frac{2nyx}{m}-2kx+\frac{n^2y^2}{mm}-\frac{2nky}{m}+kk \\ \qquad\qquad\qquad\qquad -\frac{rey}{m}+rl \end{array}\right\}=0.$$

Cum æquatio aliqua data vel inventa locum à *Parabola* determinandum includit; eandem comparo cum altera parte hujus Theorematis, nempe singulos hujus cum singulis illius terminis secundum cognitas comparationum leges, & hoc modo innotescent quantitates k, l, m, n, e, r, determinationi specialis Casus convenientes. Describenda enim est *Parabola* juxta præscriptum Constructionis istius partis Theorematis, quacum instituta fuit

fuit Comparatio, nisi quod pro quantitatibus *k*, *l*, *m*, &c. quæ in Theoremate tam quoad magnitudinem, quam positionem arbitrariæ sumuntur, assumendi sint earum valores ex comparationibus inventi, qui magnitudinem simul & positionem linearum *k*, *l*, *m*, &c. determinabunt.

Ut hoc melius intelligatur notandum, 1. Quod quantitates (m) (e) nunquam possunt esse nihilo æquales. 2. Quod m & n inveniuntur, cum inventa est earum ratio. 3. Quod m, n inventis, inventa supponitur e. 4. Quod existente $n=o$, erit $m=e$, quia tum coincidunt puncta B, C, & proinde etiam Lineæ AC, AB. 5. Quod quando valores unius aut plurium quantitatum *k*, *l*, *m*, &c. sunt negativi, tum lineæ, quas designant, ducendæ sint in partes contrarias iis, ad quas ducuntur, in constructione Theorematis; sin affirmativi sint, ad easdem, ex Algebra notum est. Atque hæc omnia de *Hyperbola* etiam & Ellipsi dicta intelligantur.

Me non latet clarissimum *Schootenium* in suis in Cartesii Geometriam Commentariis, quantitates quasdam incognitas, ex earum cum cognitis comparatione determinare. Desideratam tamen Methodi universalitatem ipsi non innotuisse constat. Primò, quod æquationis propositæ reductionem requirat. Secundò, quod æquationis sic reductæ partem extra vinculum per regulas particulares ex signis + & — dependentes construendam esse supponit, partem solummodo, quæ sub vinculo includitur, per comparationes determinans. Tertiò, quod Comparationes, quas instituit, linearum magnitudines tantùm, non verò earum positiones determinent. Neutiquam verò hæc sic acccipienda velim, quasi clarissimi Viri labores parvi faciam; ille enim finem sibi propositum egregiè assequutus est, quem non inventionem, sed *Cartesii* regularum demonstrationem hìc reddere voluisse manifestum est.

Exemp. 1. Si æquatio sit $y^2 - ax = o$, eam comparo cum prima parte Theorematis.

Erit Prima Comparatio $\frac{2n}{m} = o$, unde $n = o$, & proinde $m = e$, (per Not. 4.)

Secunda $-2k = o$, unde $k = o$.

Tertia $\frac{n^2}{m^2} = o$, sic denuo $n = o$.

Quarta

Quarta $-\frac{2nk}{m} - \frac{re}{m} = -a$, unde $r=a$.

Quinta denique Comparatio $kk+rl=o$, unde $l=o$.

Ex his habetur loci ſpecifica determinatio: nam ſecundum præſcripta prioris partis, poſitione datur vel aſſumitur linea AE, cui in angulo dato vel aſſumpto ducatur ED; jam quia $BC=n=o$, ideo lineæ AC, AF coincidunt (per not. 4.). A puncto A capiatur $AK=k=o$, ideo etiam puncta A, K, & lineæ KH, AF coincidunt. Et quia $KG=l=o$, ideo punctum G cum punctis A, K coincidunt. Itaque vertice G (ſeu A vel K) circa Diametrum GH (ſeu AE vel AF) deſcribe parabolam GD, cujus latus rectum ſit $r=a$; eritque Parabola ſic deſcripta locus quæſitus, in qua quælibet $AE=x$, $ED=y$.

Exemp. 2. Sit æquatio data $y^2-ax+bb=o$, hæc cum prima Theorematis parte comparata dabit,

Primò, $\frac{2n}{m}=o$, unde $n=o$.

Secundò, $-2k=o$.

Tertiò, $\frac{nn}{mm}=o$, ſic rurſus $n=o$, & proinde $m=e$ (not. 4.)

Quartò, $-\frac{2nk}{m} - \frac{re}{m} = -a$, unde $r=a$.

Quintò, $kk+rl=bb$, unde $l=\frac{bb}{a}$.

Ex quibus juxta præſcriptum prioris partis locus ſic determinatur. Ducatur vel poſitione detur linea indefinita AE, quacum ED faciat Angulum datum vel aſſumptum AED; jam quia $BC=n=o$, ideo puncta B, C, & lineæ AE, AF coincidunt (per not. 4.) & quia $AK=k=o$, ideo etiam puncta AK, & lineæ AF, KH coincidunt: capiatur $KG=l=\frac{bb}{a}$ (ad eaſdem partes cum Schemate conſtructionis in priori parte adhibitæ, quia valor quantitatis l eſt affirmativus per not. 5.) tum vertice G circa diametrum GH (ſeu GE vel GF) deſcribe Parabolam GD cujus parameter ſit $r=a$, dico hanc Parabolam eſſe locum æquationis propoſitæ quæſitum, in quo $AE=x$, $ED=y$.

K   *Exemp.* 3.

Exemp. 3. Sit æquatio $y^2 + ax - bb = o$, quæ cum priori parte Theorematis comparata dabit,

Primò, $\frac{2n}{m} = o$, unde $n = o$, $m = e$.

Secundò, $-2\,k = o$.

Tertiò, $\frac{nn}{mm} = o$, unde $n = o$ ut prius.

Quartò, $-\frac{2\,nk}{m} - \frac{re}{m} = a$, unde $r = -a$.

Quintò, denique $kk + rl = -bb$, unde $l = \frac{bb}{a}$.

Ex quibus juxta præſcriptum Conſtructionis in priori parte adhibitæ, habetur ſpecifica loci determinatio. Ducantur AE, ED,
Fig. 17. in angulo dato vel aſſumpto AED: jam quia BC $= n = o$ ideo coincidentibus punctis B, C, coincidunt etiam lineæ AE, ACF. Et quia AK $= k = o$, ideo etiam lineæ ACF, KH coincidunt; capiatur KG $= l = \frac{bb}{a}$, & vertice G circa diametrum GH deſcribe Parabolam GD verſus partes A tendentem, contrarias nimirum iis, ad quas ducitur in Schemate Theorematis (per not. 5.) quia valor lateris recti $r = -a$ eſt negativus; erit hæc Parabola locus quæſitus, in quo AE $= x$, ED $= y$.

Exemp. 4. Sit æquatio locum à parabola determinandum includens $xx + ay - bb = o$, quæ cum parte Theorematis ſecunda comparata dabit,

Primò, $\frac{2n}{m} = o$, unde $n = o$, $m = e$.

Secundò, $-2\,k = o$.

Tertiò, $\frac{nn}{mm} = o$, unde ut prius $n = o$.

Quartò, $-\frac{2\,nk}{m} - \frac{re}{m} = a$, unde $r = -a$.

Quintò,

Quintò, denique $kk + rl = -bb$, unde $l = \frac{bb}{a}$.

Ex quibus habetur ſpecifica loci determinatio juxta præſcriptum Conſtructionis in parte 2. adhibitæ. Ducantur AE, ED, in Angulo dato vel aſſumpto AED; quia $BC = n = o$, ideo AF, AL; & quia $AK = k = o$, ideo etiam lineæ AF, KH coincidunt: capiatur Fig. 18.
$KG = l = \frac{bb}{a}$; tum vertice G, latere recto $r = -a$ deſcribatur parabola circa diametrum GH deorſum verſus lineam AE tendens, quia valor parametri eſt negativus: erit Parabola ſic deſcripta locus quæſitus, in quo $AE = x$, $ED = y$.

Exemp. 5. Sit æquatio $y^2 - \frac{bxy}{a} + \frac{bbx^2}{4aa} - bx - dd = o$; quæ cum priori Theorematis parte comparata dabit,

Primò, $\frac{2n}{m} = \frac{-b}{a}$;

Secundò, $-2k = o$.

Tertiò, $\frac{nn}{mm} = \frac{bb}{4aa}$; jam quia ($m$) ſemper ſumi poſſit pro arbitrio (per not. 2) pono $m = a$, unde ex prima & tertia Comparatione $n = -\frac{1}{2}b$.

Quartò, $-\frac{2nk}{m} - \frac{re}{m} = -b$, unde $r = \frac{ab}{e}$.

Quintò, $kk + rl = -dd$, unde $l = -\frac{dde}{ab}$.

Ex quibus habetur Loci determinatio, juxta Conſtructionem in priori parte adhibitam. Ducantur AE, ED, in Angulo dato vel Fig. 19.
aſſumpto AED; in AE ſume $AB = m = a$; & à puncto B ducatur $BC = n = \frac{1}{2}b$ parallela ad ED, ſupra lineam AE (per not 5.) quia valor ejus eſt negativus. Per A & C ducatur linea indefinita ACF; jam quia $AK = k = o$, ideo puncta A, K, & lineæ AF, KH coincidunt; in linea KH (vel AF) capiatur $KG = l = -\frac{dde}{ab}$

 ad

ad partes ſiniſtras puncti K, quia ad partes dextras ſumitur in Schemate Theorematis, (per not. 5.) vertice G, latere recto $r = \frac{ab}{e}$ deſcribatur Parabola GD circa diametrum GH, erit hæc parabola locus quæſitus in quo AE$=x$, ED$=y$.

Exemp. 6. Sit $\left.\begin{matrix} x^2 + \frac{2byx}{a} - cx + \frac{bby^2}{aa} - \frac{bcy}{a} - \frac{1}{4}cc \\ - by \end{matrix}\right\} = 0.$

Hæc cum parte Theorematis ſecunda comparata dabit,

Primò $\frac{2n}{m} = \frac{2b}{a}$, poſito ad arbitrium $m=a$, erit $n=b$.

Secundò, $\frac{nn}{mm} = \frac{bb}{a^2}$, ut in prima.

Tertiò, $-2k=-c$, unde $k=\frac{1}{2}c$.

Quartò, $-\frac{2nk}{m} - \frac{re}{m} = -\frac{bc}{a}$, unde $r=\frac{ba}{e}$.

Quintò, $kk+rl=-\frac{1}{4}cc$, unde $l=-\frac{ecc}{2ab}$.

Ex quibus habetur loci determinatio juxta præſcriptum Conſtructionis ſecundæ partis. Ducantur itaque lineæ AE, ED in Angulo dato vel aſſumpto AED, & AL parallela ad ED, in qua capiatur AB$=m=a$, & à puncto B ducatur BC$=n=b$, parallela ad AE, per A & C ducatur indefinita linea ACF; & in AE capiatur AK$=k=\frac{1}{2}c$: à puncto K agatur KH (parallela ad AF) in qua capiatur KG$=l=-\frac{ecc}{2ab}$, ita ut G cadat infra K, quia in ſchemate ſecundæ partis G ſupra K (per not. 5.) Tum vertice G, & latere recto $r=\frac{ba}{e}$ circa diametrum GH deſcribe parabolam GD; atque hæc erit locus quæſitus in quo AE$=x$, ED$=y$.

Fig. 20.

Singulas literas ad Figuras hujus Theorematis ſpectantes, Figuræ uniuſcujuſque caſus adjeci ut facilius appareret quomodo ex eodem profluant

profluant exempla modo adducta; quæ desumpta sunt ex libro pereximio illustrissimi D. *Johannis de Witt*, qui hanc Geometriæ partem ad longe majorem perfectionem promovisset, nisi Fata cruenta Virum eripuissent de literaria Republica meritissimum.

Theor. II.

SINT x, y, *quantitates incognitæ & indeterminatæ, & fiat alterutrius harum, puta* x, *initium certum & immutabile punctum* A, *à quo per rectam positione datam* AE *indefinitè se extendere intelligatur; sitque Angulus datus vel assumptus, quem faciunt* x, y, *æqualis Angulo* AED.

CONSTRUCTIO.

Ducantur AK, BC, parallelæ ad ED, & per A, C, ducatur linea indefinita ACF, cui à puncto K parallela ducatur KH, in qua assumatur punctum aliquod G; tum vertice G circa diametrum GH describe *Hyperbolam* GD, cujus latus rectum sit GP, transversum MG, & centrum N; quantitates sic notentur. $AE = x$, $ED = y$, $AB = m$, $BC = n$, $AC = e$, $AK = k$, $KG = l$, $GN = GM = t$, $GP = r$; tum ex natura Hyperbolæ invenietur.

Theor. 2. Pars I.

$$\left.\begin{array}{l} y^2 + \frac{2nxy}{m} - 2ky + \frac{n^2x^2}{mm} - \frac{2nkx}{m} + kk \\ \quad - \frac{re^2x^2}{2mmt} + \frac{relx}{mt} + rl \\ \quad - \frac{rex}{m} - \frac{rll}{2t} \end{array}\right\} = 0.$$

Secundò, iisdem positis, *Constructio* erit ut supra: Ducatur AL parallela ad ED, à cujus puncto aliquo B ducatur BC ad AE parallela, & per A, C indefinita ACF, cui à puncto aliquo K (sumpto in linea AE) parallela sit KH; tum vertice G (in KH sumpto) circa diametrum HG describatur Hyperbola GD, cujus latus transversum GM, latus rectum GP, & centrum N, positis etiam ut supra

pra AE $=x$, ED $=y$, AB $=m$, BC $=n$, AC $=e$, AK $=k$, KG $=l$, GN $=$ NM $=t$, GP $=r$; ex eadem Hyperbolæ proprietate invenietur.

THEOR. 2. PARS 2.

$$\left.\begin{array}{r} x^2 + \frac{2nyx}{m} - 2kx + \frac{nnyy}{mm} - \frac{2nky}{m} + kk \\ - \frac{re^2yy}{2mmt} + \frac{rely}{mt} + rll \\ - \frac{rey}{m} - \frac{rll}{2t} \end{array}\right\} = 0.$$

Aſſumatur, exempli gratiâ, Carteſii Analyſis pro Quæſtione Veterum à Pappo memorata, ubi $y = m - \frac{nx}{z} + \sqrt{m^2 + ox + \frac{p}{m}xx}$: & ad confuſionem evitandam, pro m ſubſtituto c, d pro n, & b pro o; his poſitis, & æquatione ad formulam Theorematis reducta erit utique.

$$\left.\begin{array}{r} y^2 + \frac{2dxy}{z} - 2cy + \frac{ddx^2}{zz} - \frac{2dcx}{z} \\ - \frac{px^2}{c} - bx \end{array}\right\} = 0.$$

Hæc cum prima Theorematis parte comparata dabit,

Primò, $\frac{2n}{m} = \frac{2d}{z}$, & ſumpto ad arbitrium $m = z$, erit $n = d$.

Secundò, $-2k = -2c$, unde $k = c$.

Tertiò, $\frac{nn}{mm} - \frac{ree}{2m^2t} = \frac{dd}{zz} - \frac{p}{c}$; unde $t = \frac{reec}{2pzz}$.

Quartò, $-\frac{2nk}{m} + \frac{rel}{mt} - \frac{re}{m} = -\frac{2dc}{z} - b$; unde $l = \frac{reec - cezb}{2pzz}$.

Quintò,

Quintò, denique $kk + rl - \frac{rll}{2t} = o$, & ſubſtituendo valores quantitatum k, l, t, jam inventos, erit $\frac{rreec - bbczz}{4pzz} + cc = o$; quæ reducta dabit $r = \sqrt{\frac{bbz^2}{ee} - \frac{4pzzc}{ee}} \quad \frac{z}{e}\sqrt{bb - 4pc}$.

Ex quibus habetur ſpecifica loci determinatio, ſecundùm præſcripta conſtructionis in priori parte ſecundi Theorematis adhibitæ; dummodo punctum C ſupponatur in Angulo EAR. Diligenter enim hic notandum eſt, quod in priori parte horum Theorematum, lineam ED (ſeu y) ſemper ſupra lineam AE, ideoque Curvam GD ſupra diametrum GH; ſicut in poſteriori parte Curvam GD ſemper ad dextras partes diametri GH ſuppoſuerim. Sed ſi, iiſdem poſitis, hanc ad partes dextras, illam verò infra diametrum GH deſcribendam ſuppoſueris, mutanda erunt ſigna ſecundi & tertii termini Theorematis, priuſquam fiat Comparatio illius, cum propoſita qualibet æquatione: quod pariter de primo & quarto Theoremate notandum.

Vid. *Fig.* pag. 26. *Geo. Cart.*

Si diverſi proveniant valores quantitatum l, r, t, ex natura æquationis propoſitæ conſtabit, quinam ſint valores earum convenientes, qui ad locum quæſitum deſcribendum pertinebunt.

Theor. III.

S*INT Quantitates incognitæ & indeterminatæ* x, y, *Angulum facientes datum vel aſſumptum* AED.

CONSTRUCTIO.

Ducatur AK parallela ad ED, in qua ex punctis G, R, erigantur normales HGT, & RS eidem AK vel ED parallelæ, tum aſſymptotis GL, GH, deſcribatur Hyperbola FSD tranſiens per punctum S. Ponatur AE$=x$, ED$=y$, AK$=k$, KG$=l$, GR$=r$, RS$=s$; erit

THEOR. 3.

$$\left.\begin{array}{r} yx - kx + ly - lk \\ - rs \end{array}\right\} = o.$$

Ad hoc reducuntur omnes æquationes, in quibus nec xx nec yy reperiuntur, & habetur ſpecialis cujuſlibet determinatio per comparationem æquationis propoſitæ cum hoc Theoremate, ut in cæteris.

Exemp. 1. Sit $yx - bx + cy = o$ æquatio data;

Ex Prima Comparatione $+k = +b$.

Ex Secunda, $l = c$.

Ex Tertia denique, $-lk - rs = o$, unde $s = -\frac{bc}{r}$.

Et quia plures non ſuperſunt Comparationes, ideo r ad arbitrium ſumi poteſt. Ex his juxta Conſtructionem in Theoremate adhibitam Locus ſic determinatur. Ducatur AE, & ex puncto A erigatur AK$=k=b$ angulum faciens KAE æqualem Angulo quem comprehendunt x, y, per K ducatur GKL parallela ad AE, in qua capiatur ut libet GR$=r$; ex puncto R ducatur RS $= s = -\frac{bc}{r}$, (infra GL quia ejus valor eſt negativus juxta not. 5. Theor. 1.) Tum Aſſymptotis GL, HGT, deſcribatur Hyperbola FSD tranſiens per punctum S; erit hæc Hyperbola Locus quæſitus, in quo AE$=x$, ED$=y$, &c.

Quamvis quantitas (r) quoad magnitudinem ſemper ad libitum aſſumi poſſit, poſitio tamen ejus ita ordinanda eſt, ut ED (ſeu y) ſemper dextrorſum cadat ſupra lineam AE, ut conſtat ex Conſtructione in Theoremate adhibita. Ad hoc efficiendum, ita explicandus eſt valor quantitatis (s) ut pars Hyperbolæ per punctum S tranſeuntis dextrorſum cadat ſupra Lineam AE: Aſſymptotorum altera ſemper eſt GR, altera verò eſt Linea ex puncto G parallela ad RS, & ad eaſdem partes ducta.

Exemp. 2.

Exemp. 2. Sit $yx + bx - cy = o$.

Erit Prima Comparatio $k = - b$.

Secunda, $l = - c$.

Tertia, $- lk - rs = o$, unde $s = - \frac{bc}{r}$.

Ex quibus Locus quæſitus ſic deſcribitur. Ducatur linea indefinita AE; Angulum faciens datum vel aſſumptum AED; à puncto A ducatur AK$= k = - b$ parallela ad ED; à K ducatur KG$= l = - c$, & parallela ad lineam AE (ratio poſitionis utriuſque patet ex not. 5. Theor. 1.) jam in explicatione quantitatis (s) conſiderandum eſt, quo pacto pars Hyperbolæ ſupra AE exiſtat, & quidem patet hoc fieri non poſſe, niſi RS cadat ſupra KG, id eſt, niſi valor quantitatis s (*ſcil.* $- \frac{bc}{r}$) fuerit affirmativus; & quia ſumendo GR ad ſiniſtras partes puncti G, id eſt, ſumendo valorem negativum quantitatis (r), valor lineæ (s) affirmativus erit (nam $s = \frac{-bc}{-r} = \frac{cb}{r}$) concludo arbitrariam quantitatem r (=GR) ſiniſtrorſum à puncto G ſumendam eſſe; ex R ducatur RS $= \frac{bc}{r}$; & à G ducatur GH ad RS parallela; Hyperbola Aſſymptotis GK, GH, per punctum S tranſiens erit locus quæſitus, in quo AE$=x$, ED$=y$.

Theor. IV.

S*INT* x, y, *quantitates incognitæ & indeterminatæ Angulum facientes datum vel aſſumptum* AED; *ſitque* A *initium immutabile quantitatis* x *per rectam* AE *poſitione datam extenſæ.*

Ducantur AK, BC parallelæ ad ED, & per puncta A, C, linea indefinita ACF, cui à puncto K parallela ſit KH, in qua ſumatur punctum aliquod H, tum vertice G circa diametrum GH deſcribatur ſemi-ellipſis GDM, cujus tranſverſum eſt GM, latus rectum GP, ac Centrum N; lineæ verò, ut ſupra notentur, *ſcil.* AE$=x$,

 ED$=y$,

ED$=y$, AB$=m$, BC$=n$, AC$=e$, AK$=k$, KG$=l$, GN$=$MN $=t$, GP$=r$: ex natura Ellipſeos invenietur.

THEOR. 4.

$$\left.\begin{array}{l} y^2 + \frac{2nxy}{m} - 2ky + \frac{n^2x^2}{mm} - \frac{2nkx}{m} + kk \\ \qquad + \frac{reex^2}{2m^2t} - \frac{relx}{mt} + rl \\ \qquad\qquad - \frac{rex}{m} + \frac{rll}{2t} \end{array}\right\} = o.$$

Aſſumatur denuo Carteſii Analyſis, quando Locus Quæſtionem à veteribus propoſitam determinans eſt Ellipſis, *ſcilicet*, $y = c - \frac{dx}{z} + \sqrt{cc - bx - \frac{p}{c}xx}$; quæ ad formulam hujus Theorematis reducta dabit

$$\left.\begin{array}{l} y^2 + \frac{2dxy}{z} - 2cy + \frac{ddx^2}{zz} - \frac{2dcx}{z} \\ \qquad + \frac{px^2}{c} + bx. \end{array}\right\} = o.$$

Facta Comparatione hujus cum quarto Theoremate invenietur,

Primò, $\frac{2n}{m} = \frac{2d}{z}$, ſumpto ad libitum $m = z$, erit $n = d$.

Secundò, $-2k = -2c$, unde $k = c$.

Tertiò, $\frac{n^2}{m^2} + \frac{ree}{2m^2t} = \frac{dd}{zz} + \frac{p}{c}$, unde $t = \frac{reec}{2pz^2}$.

Quartò, $-\frac{2nk}{m} - \frac{rel}{mt} - \frac{re}{m} = -\frac{2dc}{z} + b$, unde $l = \frac{becz - reec}{2pzz}$,

Quintò

Quintò denique, $kk + rl + \frac{rll}{2t} = o$, unde $t = \sqrt{\frac{bbe^2 c^2 + 4 pe^2 ccc}{4 p^2 z^2}}$, $= \sqrt{\frac{bbz^2 + 4 pcz^2}{e^2}}$.

Ex quibus habetur peculiaris hujus loci determinatio, juxta conſtructionem in hoc quarto Theoremate adhibitam.

Quando Angulus datus vel Aſſumptus AED eſt talis, ut Angulus GHD ſit rectus, & ex comparationibus vel aliunde conſtet $r = 2t$, tum, Ellipſi in Circulum abeunte, planus exiſtit. Circulus enim Ellipſeos ſpecies eſt, cujus focorum diſtantia eſt nulla; vel quæ habet Tranſverſum æquale lateri recto, necnon ordinatim Applicatas ad diametrum perpendiculares.

Exemp. 2. Sit $y^2 - 2ay + x^2 = o$, ſitque Angulus quem faciunt GH, HD æqualis recto:

Erit Primò, $\frac{2n}{m} = o$, unde $n = o$, $m = e$, & quia (m) ſemper ſumi poteſt ad libitum, fiat $m = a$, unde $e = a$.

Secundò, $-2k = -2a$, unde $k = a$.

Tertiò, $\frac{nn}{mm} + \frac{ree}{2m^2 t} = 1$, unde $r = 2t$, unde conſtat Locum quæſitum eſſe Circulum.

Quartò, $-\frac{2nk}{m} - \frac{rel}{mt} - \frac{re}{m} = o$, unde $l = -t$.

Quintò, $kk + rl + \frac{rll}{2t} = o$, unde $t = a$, & proinde $r = 2a$, $l = -a$.

Ex quibus, juxta Conſtructionem in quarto Theoremate adhibitam locus ſic deſcribitur. Ducatur AE, ipſique ad Angulos rectos ED; jam quia inventum eſt BC $= n = o$, ideo puncta B, C, & Lineæ AE, AF, coincidunt; ex puncto A erigatur normalis AK $= k = a$, & per K ducatur KH utrinque indefinita, & parallela ad

ad AF (ſeu AE), & fiat KG$=l=-a$; & quia GN$=$MN$=t=a$, ideo GM$=2t=2a$, coincidentibus proinde punctis K, N. Centro itaque N, latere tranſverſo GM$=2t=2a$, & latere recto $r=2a$ deſcribatur Ellipſis; hoc eſt centro N (ſeu K) & radio NG$=$NK$=$NM$=a$ deſcribatur Circulus AGDMO, atque hic erit locus quæſitus, in quo AE$=x$, ED$=y$, & EO$=y$. Ex primis enim Algebræ elementis notum eſt propoſitam Æquationem duas habere veras Radices.

FINIS.
